Optimal Execution under Liquidity Uncertainty

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Joint work with Etienne Chevalier, Yadh Hafsi and Vathana Ly Vath

Quantitative Finance at La Défense

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Introduction

The Execution Problem:

- An agent must execute $\overline{X}$ number of shares of a financial asset by time $T < +\infty$.
- The agent is a trader with continuous observations of the Limit Order Book (LOB).
- Both continuous and discrete trading are allowed.

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Focus:

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Liquidity Dynamics: Market Impact

- Fix a liquidity regime i
- Impact function derived from the limit order book (LOB) shape F_i.
- Quantity at distance $x \geq 0$ from fundamental price A_t for a continuous LOB: $F_i(x) := \int_0^x f_i(p)\mathrm{d}p$.
- Price impact for order size y: $\psi_i(y) := \sup\{a \geq 0 : F_i(a) < y\}$, with $\psi_i(0) = 0$.
- Post-trade price: $A_t + \psi_i(y)$.

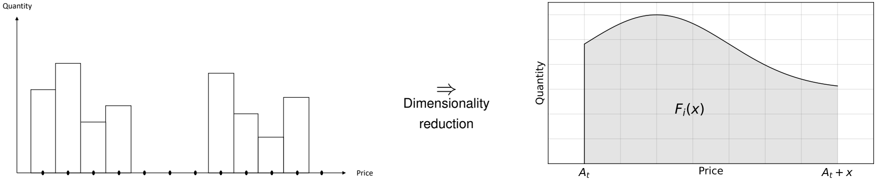

Figure 1: Illustration of a limit order book density and resulting price impact.

Liquidity Dynamics: Price Modeling and Volume Effect

- **Liquidity Regimes I:** A stable and conservative Markov chain $\{I_t\}_{t\geq 0}$ on finite state space $E = \{1, \dots, d\}$.
- **Volume effect** Y: an $\mathscr{F}$-adapted nonnegative process, such that for all $t \geq 0$ and $u \in [t, T]$,

$$\begin{cases} \mathrm{d}Y_u = -h(Y_{u^-})\mathrm{d}u + \sigma(Y_{u^-})\mathrm{d}W_u + \int_{\mathbb{R}} q(Y_{u^-}, z) M(\mathrm{d}u, \mathrm{d}z), \text{ (comp. of } M(du,dz) = \lambda_u \nu(dz)du) \\ Y_{t^-} = y. \end{cases}$$

- **Price P:** $P_t := \underbrace{A_t}_{\text{fundamental price}} + \underbrace{D_t}_{\text{price deviation}}$, with $D_t := \psi_{I_t}(Y_t)$, for all $t \geq 0$.

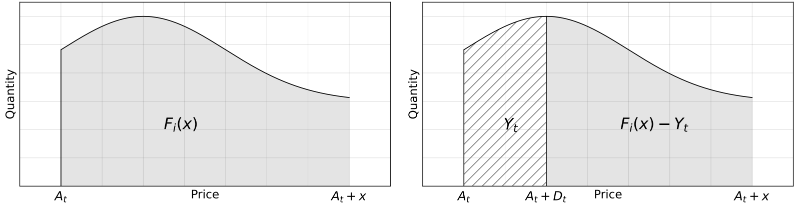

Figure 2: Representation of a limit order book at time t. The left-hand figure showing the state before a transaction and the right-hand figure showing it after. The dashed area Y_t denotes consumed shares.

Problem Formulation

Objective:

→ Purchase a position $\bar{X}$ within a finite time horizon T, starting from $X_{t-} = x$ if $I_{t-} = i$ and $Y_{t-} = y$.

- Supposing $y = 0$, the purchase cost for a large trade of size Δy, in excess of A_t at time t is equal to

$$\Phi_i(\Delta y) := \int_0^{\psi_i(\Delta y)} \xi \, \mathrm{d}F_i(\xi) = \int_0^{\Delta y} \psi_i(\zeta)\, \mathrm{d}\zeta, \quad \forall (y,i) \in \mathbb{R}_+ \times \mathbb{I}_m.$$

- The total cost is

$$\pi_t^i(\Delta y) = \int_0^{\Delta y} \left[A_t + \psi_i(\zeta)\right] \mathrm{d}\zeta = \underbrace{A_t \Delta y}_{\text{Cost at the current price}} + \underbrace{\Phi_i(\Delta y)}_{\text{Impact cost}} .$$

- An admissible purchase strategy consists of a non-decreasing $\mathscr{F}$-adapted right-continuous process $X = (X_u)_{t \le u \le T}$ such that $X_{t^-} = x$ and $X_T = \bar{X}$.

Optimal Execution: Value Function

Controlled Dynamics:

$$\begin{cases} \mathrm{d}Y_u^{t,y,X} = \mathrm{d}X_u - h(Y_{u^-}^{t,y,X})\mathrm{d}u + \sigma(Y_{u^-}^{t,y,X})\mathrm{d}W_u + \int_{\mathbb{R}} q(Y_{u^-}^{t,y,X}, z)M(\mathrm{d}u, \mathrm{d}z), \\ Y_{t^-}^{t,y,X} = y, \end{cases}$$

with $u \in [t, T]$ and $y \geq 0$.

Value Function:

- Minimize the costs over admissible strategies $X \in \mathscr{A}_t(x)$:

$$v(i,t,x,y) = v_i(t,x,y) := \inf_{X \in \mathscr{A}_t(x)} \mathbb{E}\Big[\int_t^T \psi_{I_u}(\check{Y}_{u^-}^{t,y,X})\mathrm{d}X_u^c + \sum_{t \leq u \leq T} \Phi_{I_u}(Y_u^{t,y,X}) - \Phi_{I_u}(\check{Y}_{u^-}^{t,y,X})\Big],$$

  for all $(t,x,y) \in \mathscr{S} := [0,T[\times [0, \bar{X}[\times \mathbb{R}_+^*$ and $i \in \mathbb{I}_m$.

→ *Boundary conditions:* $v_i(T,x,y) = \Phi_i(y + \bar{X} - x) - \Phi_i(y)$ and $v_i(t, \bar{X}, y) = 0$.

→ *Growth condition:* $0 \leq v_i(t,x,y) \leq \Phi_i(y + \bar{X} - x) - \Phi_i(y)$.

Optimal Execution: Dynamic Programming Principle and HJBQVI

Value Function and HJBQVI:

- v solves the Hamilton-Jacobi-Bellman Quasi-Variational Inequality (HJBQVI):

$$\max\Big(-\frac{\partial v_i}{\partial t} - \mathscr{L} v_i - \sum_{j\neq i}(v_j - v_i)Q_{ij}, -\frac{\partial v_i}{\partial x} - \frac{\partial v_i}{\partial y} - \psi_i\Big) = 0, \text{ on } \mathscr{S}.$$

  where $\mathscr{S} := [0,T]\times[0,\bar{X}]\times\mathbb{R}_+$. → **viscosity solutions**

- The partial integro-differential operator $\mathscr{L}$ is given by

$$\mathscr{L}\varphi := \frac{1}{2}\sigma^2(y)\frac{\partial^2\varphi}{\partial y^2} - h(y)\frac{\partial\varphi}{\partial y} + \lambda_t\int_{\mathbb{R}}(\varphi(t,x,y+q(y,z)) - \varphi)\,\nu(\mathrm{d}z),$$

Theorem 1 (Dynamic Programming Principle and HJBQVI)

The value function v is a continuous function on $\overline{\mathscr{S}}$. For any stopping time τ in $[t,T]$, $(t,x,y)\in\mathscr{S}$ and $i\in\mathbb{I}_m$, we have

$$v_i(t,x,y) = \inf_{X\in\mathscr{A}_t(x)}\mathbb{E}\Big[\int_t^\tau \psi_{I_{u^-}}(\check{Y}^{t,y,X}_{u^-})\mathrm{d}X^c_u + \sum_{t\le u\le\tau}\big(\Phi_{I_u}(Y^{t,y,X}_u) - \Phi_{I_u}(\check{Y}^{t,y,X}_{u^-})\big) + v_{I_\tau}(\tau, X_\tau, Y^{t,y,X}_\tau)\Big].$$

The value function v is the unique viscosity solution of the HJBQVI.

The Free Boundary Problem

Definition 3.1

We define the exercise region $\mathscr{E}_i := \overline{int(\mathscr{E}_i^{diff})}$ *as the closure of interior of the set* $\mathscr{E}_i^{diff}$*, on which* v_i *is differentiable, with*

$$\mathscr{E}_i^{diff} := \{(t,x,y) \in \mathscr{S} \setminus \mathscr{N}_i : -\frac{\partial v_i}{\partial x} - \frac{\partial v_i}{\partial y} - \psi_i = 0\},$$

and we define the continuation region $\mathscr{C}_i := \mathscr{S} \setminus \mathscr{E}_i$ *as its complement.*

Proposition 3.1 (Connectedness)

Assume that the interiors of $\mathscr{C}_i$ *and* $\mathscr{E}_i$ *are non-empty, and that* $\sigma + \lambda > 0$*. Then, for each* $i \in \mathbb{I}_m$*, the free boundary* $\partial \mathscr{E}_i$ *is non-empty and path-connected. Moreover, if* $(t,x,y) \in \mathscr{E}_i$*, then*

$$(t,x,y') \in \mathscr{E}_i \quad \text{for all } 0 \le y' \le y, \quad \text{and} \quad (t,x',0) \in \mathscr{E}_i \quad \text{for all } 0 \le x' \le x.$$

Illustration of the boundary

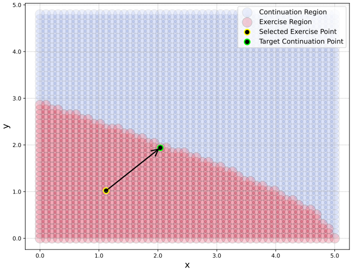

Figure 3: Illustration of continuation (blue) and exercise (red) regions, plotted against purchased quantity (x-axis) and volume effect (y-axis). The arrow shows an impulse shifting a state from exercise to continuation along $y = x$.

Numerical Examples: Parameterization

- Volume effect:

$$h(y) = cy, \quad \sigma(y) = dy, \quad \text{and} \quad q(y,z) = eyz, \quad \forall y,z \in \mathbb{R}_+,$$

- Order volumes follow an exponential distribution: $\nu(\mathrm{d}z) = \eta e^{-\eta z} \mathbb{1}_{\{z>0\}} \mathrm{d}z, \quad \forall z \in \mathbb{R}.$
- Limit order book shape and impact functions:

$$\psi_i(y) = \begin{cases} \frac{y}{\kappa}, & \text{if } \gamma_i = 0, \\ e^{\frac{y}{\kappa}} - 1, & \text{if } \gamma_i = 1, \\ \left[1 + (1-\gamma_i)\frac{y}{\kappa}\right]^{\frac{1}{1-\gamma_i}} - 1, & \text{else.} \end{cases}$$

- Default parameter values:

Parameter	c	d	e	η	κ	γ_0	λ	$\bar{X}$	$\bar{Y}$	T
Value	0.5	0.1	0.2	1.0	0.8	-1	0.5	4	10	4.0

Numerical Results: Single Regime Case

→ Impact of drift h, jump intensity λ, and volatility σ on the exercise and continuation regions.

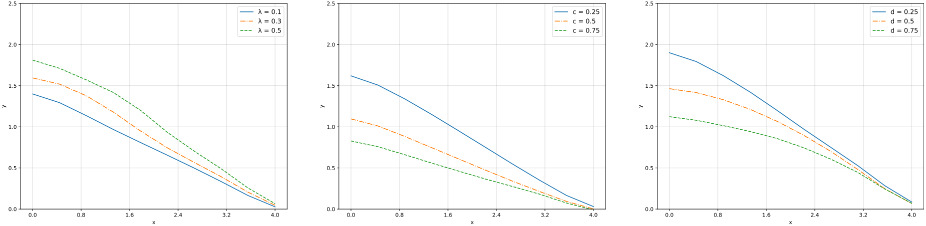

Figure 4: Variation of the exercise boundary $x \mapsto y^*(T/2, x)$ under different market conditions with jump intensity on the left, resilience in the middle, and volatility on the right.

Numerical Results: Regime-Switching case

We assume a two-state homogeneous Markov chain, with the transition rate matrix $Q(t)$ at time $t \in [0, T]$ given by

$$Q(t) = \begin{pmatrix} -q_1 & q_1 \\ q_2 & -q_2 \end{pmatrix}.$$

We fix $q_1 = q_2 = 0.2$ and $\gamma_1 = 0$. With these parameters set, we then vary γ_2.

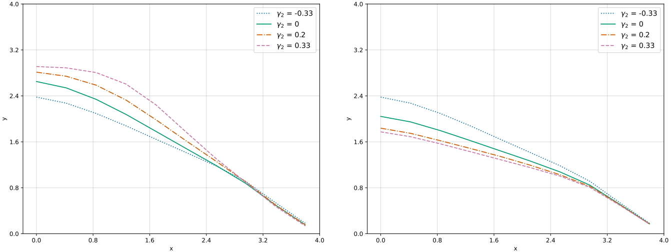

Figure 5: Variation of the exercise boundary $x \mapsto y^*(T/2, x)$ of v_0 on the left and v_1 on the right under different price impacts.

Conclusions

- Developed a stochastic control framework for optimal execution under general price impact and stochastic liquidity, modeled via a jump diffusion volume effect and regime-switching Markov processes.
- Characterized the value function as the unique viscosity solution of a system of HJBQVIs. Proved regularity properties of the value function, including continuity, monotonicity, and connectedness of the free boundary.
- Approximated the solution to quantify the effect of liquidity uncertainty on optimal strategies.

Thank You !

Chevalier, E., Hafsi, Y., Ly Vath, V., Pulido, S. (2026). Optimal Execution under Liquidity Uncertainty. arXiv:2506.11813. To appear in SIAM Journal on Financial Mathematics.

Numerical Results: Single Regime Case

→ Impact of drift *h*, jump intensity λ, and volatility σ on the execution costs (i.e, value function).

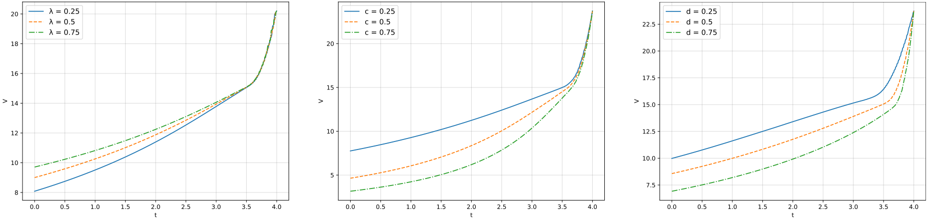

Figure 6: Variation of the value function over time under different market conditions with jump intensity on the left, resilience in the middle, and volatility on the right.

Numerical Results: Regime-Switching case

→ The price impact parameter is set to $\gamma_1 = 0$ in regime 1 and $\gamma_2 = 0.5$ in regime 2. We consider $q_1 = q_2 = q_0$ and vary the values of q_0. The probability of being in either regime is equal, but the frequency of transitions between regimes changes.

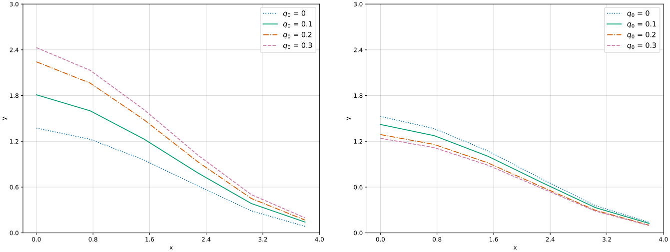

Figure 7: Variation of the exercise boundary $x \mapsto y^*(T/2, x)$ of v_0 on the left and v_1 on the right under different regime switching intensities.

Numerical Results: Regime-Switching case

→ The price impact parameter is set to $\gamma_1 = 0$ in regime 1 and $\gamma_2 = 0.5$ in regime 2. We fix $q_1 = 0.2$ and vary q_2, and study how the asymmetry in switching probabilities affects execution.

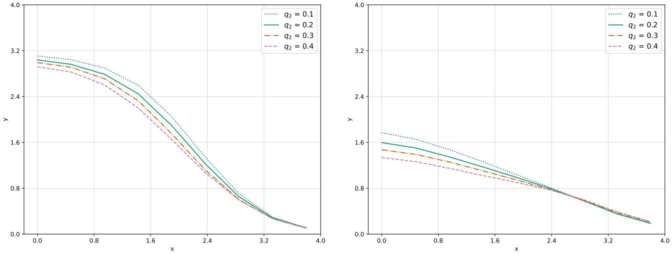

Figure 8: Variation of the exercise boundary $x \mapsto y^*(T/2, x)$ of v_0 on the left and v_1 on the right under different asymmetric regime switching intensities.

Motivation

→ Stochastic liquidity and regime changes.

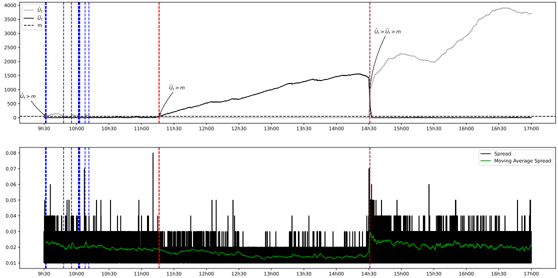

Figure 9: Changes in the spread observed in BNP stock data on 11/05/2022, Chevalier, Hafsi, and Ly Vath (2023).

Liquidity Dynamics: Price Modeling and Volume Effect

Assumptions 4.1

(A1) *There exists $b > 0$, $\beta > 0$ and $a > 0$ such that*

$$F_i(x) \geq bx^{\beta}, \quad \forall (x,i) \in [a, +\infty[\times \mathbb{I}_m.$$

(A2) *The measure $\nu(\mathrm{d}z)$ satisfies*

$$\int_{\mathbb{R}} (1+z^2)\nu(\mathrm{d}z) < +\infty,$$

and $y \mapsto y + q(y,z)$ is non-decreasing for every z

(A3) *There exists a constant $C > 0$ such that, for all $y \geq 0$,*

$$|h(y)| + |\sigma(y)| + \int_{\mathbb{R}} |q(y,z)|\nu(\mathrm{d}z) \leq C(1+|y|).$$

(A4) *There exists a constant $L > 0$ such that, for all $y, y' \geq 0$,*

$$|h(y) - h(y')| + |\sigma(y) - \sigma(y')| + \int_{\mathbb{R}} |q(y,z) - q(y',z)|\nu(\mathrm{d}z) \leq L|y - y'|.$$

Analytical Properties of the Value Function

Proposition 4.1

The value function v is finite.

Proposition 4.2 (Monotonicity)

For any $(i,t,x,y) \in \mathbb{I}_m \times \overline{\mathscr{S}}$, the following results hold:

1. $t_0 \mapsto v_i(t_0,x,y)$ *is non-decreasing on* $[0,T]$.
2. $x_0 \mapsto v_i(t,x_0,y)$ *is non-increasing on* $[0,\bar{X}]$.
3. $y_0 \mapsto v_i(t,x,y_0)$ *is non-decreasing on* $\mathbb{R}_+$.

Theorem 4.1

The value function v is a continuous function on $\overline{\mathscr{S}}$.

Analytical Properties of the Value Function

Sketch of proof: Compare the sample paths of $Y^{t,y,X}$ and $Y^{t,y',\widetilde{X}}$. Design a strategy that ensures the paths of $u \mapsto Y_u^{t,y,X}$ and $u \mapsto Y_u^{t,y',\widetilde{X}}$ meet at a specific point in time τ, facilitating their comparison and enabling us to establish bounds for $v_i(t,x,y') - v_i(t,x,y)$.

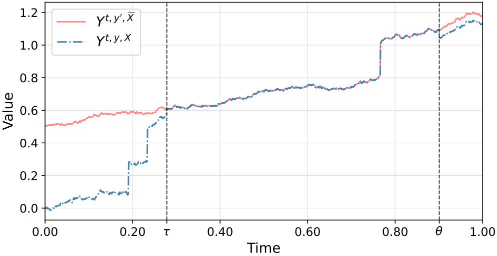

Figure 10: Illustration of $Y^{t,y,X}$ and $Y^{t,y',\widetilde{X}}$ sample paths in the scenario where $y < y'$, $\tau < \theta < T$ and $\Delta X_\tau + Y_\tau^{t,y,X} - Y_\tau^{t,y',\widehat{X}} < \overline{X} - \widehat{X}_{\tau-}$.

Viscosity Characterization

Definition 4.1 (Viscosity Solution)

We define a viscosity solution of HJBQVI as follows :

1. *v is a continuous viscosity supersolution (resp. subsolution) of HJBQVI on $\mathbb{I}_m \times \mathscr{S}$ if it satisfies the growth conditions, and if*

$$\max\Big(-\big(\frac{\partial\varphi}{\partial t} + \mathscr{L}\varphi + \sum_{j\neq i}(v_j - v_i)Q_{ij}\big)(t,x,y),\ -\big(\frac{\partial\varphi}{\partial x} + \frac{\partial\varphi}{\partial y} + \psi_i\big)(t,x,y)\Big) \geq (\text{resp. } \leq) 0,$$

   for any $(i,t,x,y) \in \mathbb{I}_m \times \mathscr{S}$ and any smooth test function $\varphi \in C^{1,2}(\mathscr{S})$ such that $(v_i - \varphi)$ attains a local minimum (resp. maximum) at (t,x,y) over the set $[t, t+\delta[\times [x, x+\delta[\times B_\delta(y) \subset \mathscr{S}$ for some $\delta > 0$, with $(v_i - \varphi)(t,x,y) = 0$.

2. *v is a continuous viscosity solution on $\mathbb{I}_m \times \mathscr{S}$ if it is both a viscosity supersolution and subsolution of the HJBQVI.*

Viscosity Characterization

Theorem 4.2

The value function v is a viscosity subsolution of the HJBQVI.

Theorem 4.3

The value function v is a viscosity supersolution of HJBQVI.

Theorem 4.4 (Strong Comparison Principle)

If v_i is a continuous viscosity subsolution and w_i is a continuous viscosity supersolution of the HJBQVI, such that

$$v_i(t,\bar{X},y) \leq w_i(t,\bar{X},y), \;\; \textit{and} \;\; v_i(T,x,y) \leq w_i(T,x,y),$$

for all $(i,t,x,y) \in \mathbb{I}_m \times \overline{\mathscr{S}}$, then $v_i \leq w_i$ on $\mathscr{S}$.

Strong comparison principle

Sketch of proof: Let $\beta > 0$ such that

$$\lim_{y\to+\infty} \max_{i\in I_m} \frac{\Phi_i(y+\bar{X}-x)-\Phi_i(y)}{y^\beta} = 0, \quad \forall (x,y)\in[0,\bar{X}]\times\mathbb{R}_+.$$

Define $\varphi_i : \overline{\mathscr{S}} \to \mathbb{R}$ such that

$$\varphi_i(t,x,y) := -e^{-ct}\big((-a_1x+a_2)y^\beta - b_1x+b_2\big), \quad \forall (t,x,y)\in\overline{\mathscr{S}},$$

where a_1, a_2, b_1, b_2 and c are positive constants. Define $v_{i,m} : \overline{\mathscr{S}} \to \mathbb{R}$ such that

$$v_{i,m} := v_i + \frac{1}{m}\varphi_i.$$

$\Rightarrow$ $v_{i,m}$ is a strict subsolution. We get a contradiction using *Ishii's lemma*.

The Free Boundary Problem

Sketch of proof: Assume that $\mathring{\mathscr{C}}_i$ and $\mathring{\mathscr{E}}_i$ are non-empty. Let $z_0 := (t_0, x_0, y_0) \in \mathring{\mathscr{E}}_i$ be an interior point of $\mathscr{E}_i$. Suppose there exists $\delta > 0$ one of the following options,

$$\mathscr{O}_c := \{(t_0, x_0, y) \in \overline{\mathscr{S}} : y + \delta < y_0\} \subset \mathscr{C}_i, \ \text{and} \ \mathscr{O}_e := \{(t_0, x_0, y) \in \overline{\mathscr{S}} : y_0 < y + \delta\} \subset \mathscr{E}_i,$$

$$\mathscr{O}_c := \{(t_0, x, 0) \in \overline{\mathscr{S}} : x + \delta < x_0\} \subset \mathscr{C}_i, \ \text{and} \ \mathscr{O}_e := \{(t_0, x, 0) \in \overline{\mathscr{S}} : x_0 < x + \delta\} \subset \mathscr{E}_i,$$

where $\mathscr{O}_c$ is non-empty segment.

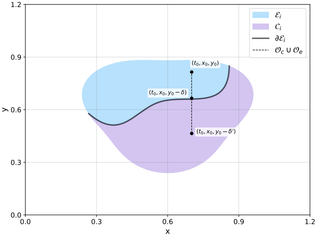

Figure 11: Illustration of $\mathscr{O}_e$, $\mathscr{O}_c$ and $\widetilde{\mathscr{O}}$.

The Free Boundary Problem

Sketch of proof: Construct a test function $\varphi_\varepsilon \in C^2(\mathscr{S})$ such that z_0 achieves a local maximum of $v_i - \varphi_\varepsilon$, $v_i(z_0) = \varphi_\varepsilon(z_0)$ and $\lim_{\varepsilon\to 0} -\mathscr{L}\varphi_\varepsilon(z_0) = +\infty$. Since v_i is a viscosity subsolution of the HJBQVI, we get

$$\max\Big(-\big(\frac{\partial\varphi_\varepsilon}{\partial t} + \mathscr{L}\varphi_\varepsilon + \sum_{j\neq i} Q_{ij}(v_j - \varphi_\varepsilon)\big)(z_0),\ -\big(\frac{\partial\varphi_\varepsilon}{\partial x} + \frac{\partial\varphi_\varepsilon}{\partial y} + \psi_i\big)(z_0)\Big) \leq 0.$$

The constructed funciton φ_ε satisfies $\lim_{\varepsilon\to 0} -\mathscr{L}\varphi_\varepsilon(z_0) = +\infty$, while

$$-\big(\mathscr{L}\varphi_\varepsilon + \sum_{j\neq i} Q_{ij}(v_j - v_i)\big)(z_0) \leq 0$$

which leads to a contradiction.