From rough to multifractal multidimensional volatility: a multidimensional Log S-fBM model

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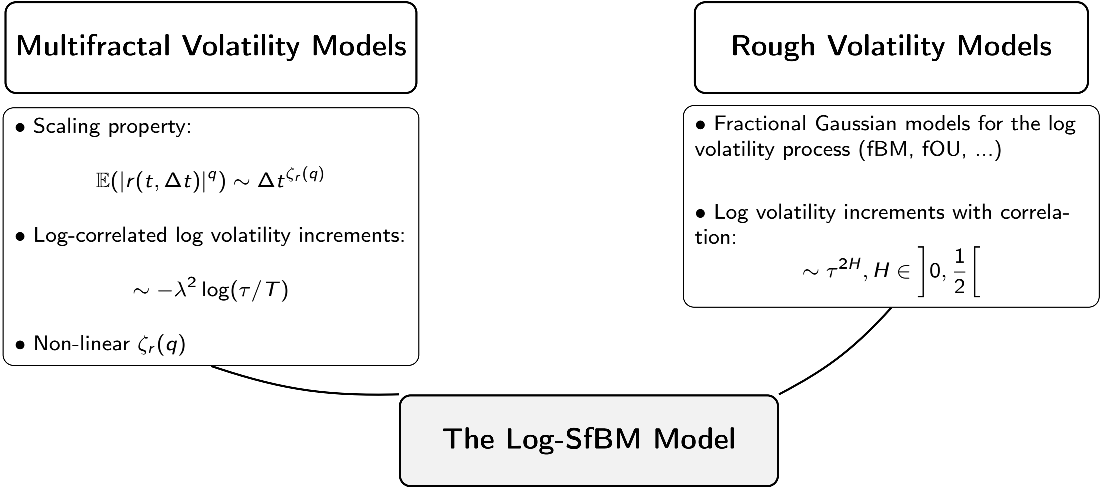
Overview

1. **Introduction and context**
2. **The Log S-fBM model**
3. **The multidimensional Log S-fBM model**
4. **Conclusion**

Introduction and context

Introduction: The Log S-fBM model unifies rough and multifractal volatility



Multifractal Volatility Models

- Scaling property:

$$\mathbb{E}(|r(t,\Delta t)|^q) \sim \Delta t^{\zeta_r(q)}$$

- Log-correlated log volatility increments:

$$\sim -\lambda^2 \log(\tau/T)$$

- Non-linear $\zeta_r(q)$

Rough Volatility Models

- Fractional Gaussian models for the log volatility process (fBM, fOU, ...)
- Log volatility increments with correlation:

$$\sim \tau^{2H}, H \in \left]0, \frac{1}{2}\right[$$

The Log-SfBM Model

The Log S-fBM model

The S-fBM process

Proposition [Wu *et al.*(2022)]

A ***S-fBM process*** $\omega_{H,T}$ *is a stationary gaussian process with:*

$$\forall (t,\tau) \in \mathbb{R}_+, \quad \mathrm{Cov}(\omega_{H,T}(t+\tau), \omega_{H,T}(t)) = \frac{\lambda^2}{2H(1-2H)} \left(1 - \left(\frac{\tau}{T}\right)^{2H}\right) \mathbb{1}_{\{|\tau| \le T\}}. \quad (1)$$

where $H \in \left]0, \frac{1}{2}\right[$.

Key properties:

- Fractional regularity controlled by H.
- λ^2 is the intermittency parameter, identifiable when $H \sim 0$ as $\lambda^2 \sim 2H Var\left(\omega_{H,T}(t)\right)$

The Log S-fBM model: Unifying rough and multifractal volatility

The Log S-fBM random measure $M_{H,T}$ is defined as: $M_{H,T}(\mathrm{d}t) = \exp(\omega_{H,T}(t))\mathrm{d}t$.

Proposition [Wu *et al.*(2022)]

There exists a multifractal random measure $\tilde{M}_T(\mathrm{d}t)$ such that:

$$M_{H,T}(\mathrm{d}t) \xrightarrow[H \to 0]{\mathrm{w}} \tilde{M}_T(\mathrm{d}t) \tag{2}$$

This allows to derive a stochastic volatility model, the **Log S-fBM model** valid in the rough and multifractal regimes:

Definition [Wu *et al.*(2022)]

*The **Log S-fBM model** is of the form* $dX_t = \sqrt{\frac{M_{H,T}(\mathrm{d}t)}{\mathrm{d}t}} dB_t$, *where* $M_{H,T} \perp\!\!\!\perp B$.

The small intermittency approximation

By definition: $\forall I \subset \mathbb{R}_+, M_{H,T}(I) := \int_I e^{\omega_{H,T}(t)} dt$ and considering $\Omega_{H,T}(I) = \frac{1}{\lambda} \int_I (\omega_{H,T}(u) - \mathbb{E}[\omega_{H,T}(u)])\, du$.

Proposition [Wu *et al.*(2022)]

Let $t_1, \dots, t_n$ be n arbitrary times and $\Delta > 0$. The following holds:

$$\mathbb{E}\left[\ln\left(\frac{M_{H,T,\Delta}(t_1)}{\Delta}\right) \cdots \ln\left(\frac{M_{H,T,\Delta}(t_n)}{\Delta}\right)\right] = \lambda^n \Delta^{-n} \mathbb{E}\left[\Omega_{H,T,\Delta}(t_1) \cdots \Omega_{H,T,\Delta}(t_n)\right] + o(\lambda^n), \tag{3}$$

where $\Omega_{H,T,\Delta}(t) = \Omega_{H,T}([t, t+\Delta])$ and $M_{H,T,\Delta}(t) = M_{H,T}([t, t+\Delta])$.

Given this result, the calibration is performed through the so called **Generalized method of moments (GMM)**.

A particular multivariate Log S-fBM model: The Nested stationary fractional factor model (N-SfFM) (Zarhali *et al.*)

- Zarhali *et al.* propose as a multivariate stochastic volatility model using the Log S-fBM process:

$$\forall i \in \{1, .., N\}, \quad \begin{cases} dx_t^i = \beta_i df_t + de_t^i \\ df_t = e^{\frac{\Omega_t}{2}} dB_t \\ de_t^i = \sigma_i e^{\frac{\gamma_i \Omega_t + \omega_t^i}{2}} dB_t^i \end{cases}$$

- We build upon the intuitions behind the Nested factor model of Bouchaud & Chicheportiche(2015), reproducing:
    - A non elliptical joint distribution of asset returns
    - Non linear dependences between assets

The multidimensional Log S-fBM model

The multidimensional S-fBM process (mS-fBM)

From now on, we consider the two symmetric matrices:

- $\boldsymbol{H} := (H_{i,j})_{1 \le i,j \le d} \in \left]0, \frac{1}{2}\right[^{d \times d}$ the coHurst matrix
- $\boldsymbol{\xi} := (\xi_{i,j})_{1 \le i,j \le d} \in \mathbb{R}^{d \times d}$ namely the cointermittency matrix with diagonals $\boldsymbol{\xi}_{i,i} := \lambda_i^2$

such that:

$$\boldsymbol{\mathcal{H}}_1 : \quad \boldsymbol{\xi} \text{ is positive definite} \tag{4}$$

$$\boldsymbol{\mathcal{H}}_2 : \quad \boldsymbol{H} \text{ is conditionally negative definite, i.e: } \forall c \in \mathbb{R}^d, \sum_{i,j=1}^{d} c_i c_j H_{ij} \le 0, \text{ such that } \sum_{i=1}^{d} c_i = 0 \tag{5}$$

are satisfied.

mS-fBM process

Proposition [Zarhali *et al.*(2026)]

A ***mS-fBM process*** $((\omega_i(t))_t, i \in [\![1,d]\!])$ *is stationary and gaussian with:*

$$\forall (i,j) \in [\![1,d]\!]^2, \qquad \mathrm{Cov}(\omega_i(t), \omega_j(s)) \sim \xi_{i,j} \left[C_{i,j} - \left(\frac{\tau}{T}\right)^{2H_{i,j}} D_{i,j} \right] \mathbb{1}_{\{\tau \leq T\}}. \tag{6}$$

where $\tau = |t-s|$, $C_{i,j}$ *and* $D_{i,j}$ *depend on* $H_{i,j}$ *and* $\overline{H}_{i,j} := \frac{H_{i,i}+H_{j,j}}{2}$.

⇒ **As in 1D,** $H_{i,j}$ **represents the decorrelation speed parameter.**

mLog S-fBM model: Unifying rough and multifractal multidimensional volatility

Let $\boldsymbol{M_H}$, an $\mathbb{R}^d$ valued random measure defined as:

$$(\boldsymbol{M_H}(dt))_j = \begin{cases} M_j(dt), \text{if } H_{j,j} \neq 0 \\ \widetilde{M}^j(dt), \text{if } H_{j,j} = 0 \end{cases} \tag{7}$$

$M_i(dt) := \exp(\omega_i(t))\, dt$ and $\widetilde{M}^j$ is the i^{th} marginal of a $\mathbb{R}^d$ valued multifractal random measure $\widetilde{\boldsymbol{M}}$.

Theorem [Weak continuity and limit $\|H\| \to 0$, Zarhali *et al.*(2026)]

- *The sequence of measures* $(\boldsymbol{M}_{\boldsymbol{H}_n})_n$ *converges weakly to* $\boldsymbol{M_H}$ *as* $\|\boldsymbol{H}_n - \boldsymbol{H}\| \to 0$.
- *The measure* $\boldsymbol{M_H}$ *converges weakly to* $\widetilde{\boldsymbol{M}}$ *as* $\|\boldsymbol{H}\| \to 0$.

⇒ **This unifies multidimensional rough and multifractal volatility.**

mLog S-fBM process and the small intermittency approximation

Definition [Zarhali *et al.*(2026)]

A ***mLog S-fBM*** *is the* $\mathbb{R}^d$ *valued process* $(\mathbb{X}_t)_t := ((X_t^i)_t, i \in [\![1, d]\!])$ *with:*

$$\begin{cases} dX_t^i = \sqrt{\frac{(\boldsymbol{M_H}(\mathrm{d}t))_i}{\mathrm{d}t}} dB_t^i, \\ X_0^i = x_0^i. \end{cases} \tag{8}$$

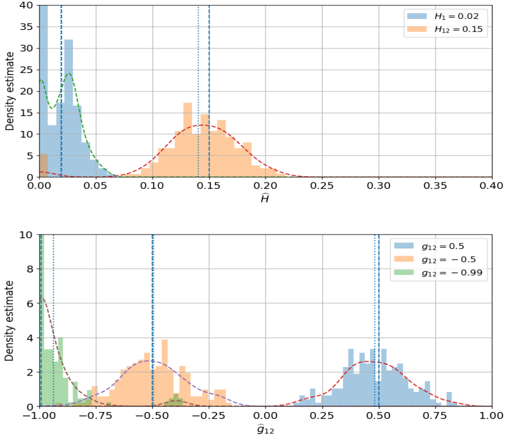
where $((B_t^i)_t, i \in [\![1, d]\!]) \perp\!\!\!\perp \boldsymbol{M_H}$.

Corollary [Small intermittency approximation, case d=2]

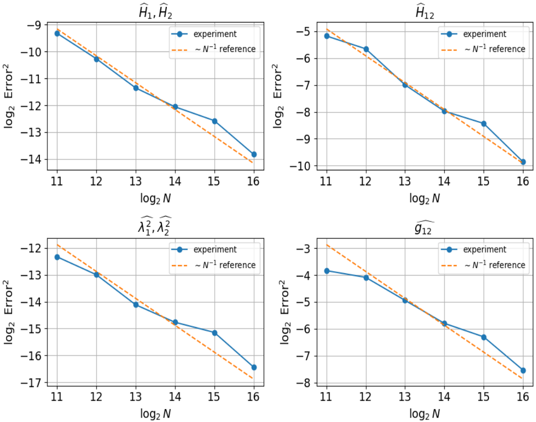
$$\forall (I, J) \subset \mathbb{R}_+^2, \quad \mathbb{E}\left(\ln\left(\frac{M_i(I)}{|I|}\right) \ln\left(\frac{M_j(J)}{|J|}\right)\right) = \lambda_i \lambda_j \mathbb{E}\left(\frac{\Omega_i(I)}{|I|} \frac{\Omega_j(J)}{|J|}\right) + o\left(||\Lambda_{i,j}||^2\right) \tag{9}$$

where $(i, j) \in [\![1, d]\!]$, $\Lambda_{i,j} = \begin{pmatrix} \lambda_i \\ \lambda_j \end{pmatrix}$ *and* $\Omega_i(I) = \frac{1}{\lambda_i} \int_I (\omega_i(u) - \mathbb{E}[\omega_i(u)]) \, du$

Estimation Robustness

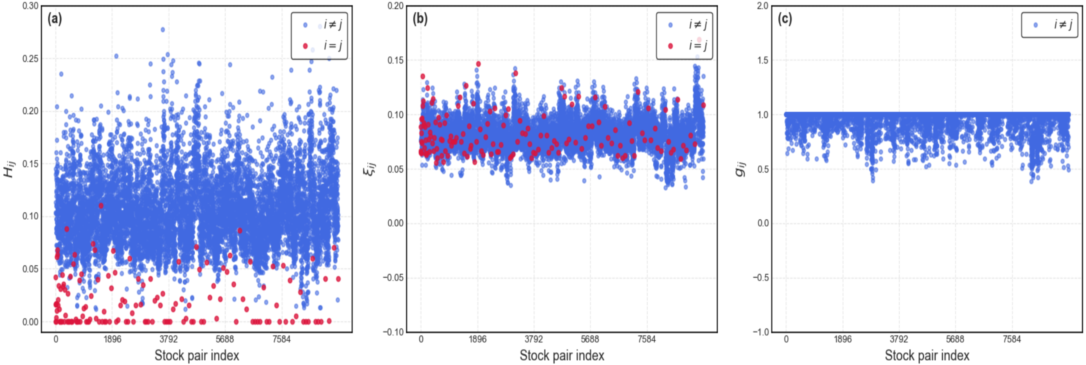


Empirical distributions of $H_{i,j}$ (**top**) and $g_{i,j}$ (**bottom**) from 200 synthetic samples.



(**Blue**) Log-standard deviation of the calibrated parameters; (**orange**) corresponding linear fits.

Calibration on S&P500 data



Scatter plots for all the selected assets of the S&P500 of: the Hursts and coHurst exponents **(a)**, the intermittency and cointermittency coefficients **(b)** and the cointermittency correlation coefficients **(c)**.

Conclusion and research directions

The purpose of this talk was to demonstrate the construction of multivariate dynamics using the Log S-fBM model:

- The **mLog S-fBM** is richer and tractable with a systematic calibration method
- Many directions remain: understand the structure of the co-Hurst matrix, investigate the correlation scale invariance ...

Thanks for you attention

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