Optimal Control, Entropy, and Market Making (To Say Nothing of the Quantum Harmonic Oscillator)

Prof. Olivier Guéant

Laboratoire de Probabilités, Statistique et Modélisation (LPSM)
Université Paris-Cité

Quantitative Finance at La Défense 2026, June 2026

What is Market Making?

Role and Business of a Market Maker

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- Problem orthogonal to those of traditional financial mathematics, *e.g.*, pricing problems.
- It has applications beyond financial markets (*e.g.*, second-hand markets – think Gibert, Momox, etc.).

My Entry Point (around 2010)

The financial mathematics community got interested in market making from 2008 following the paper by Avellaneda and Stoikov.

High-frequency trading in a limit order book

MARCO AVELLANEDA and SASHA STOIKOV*

Mathematics, New York University, 251 Mercer Street, New York, NY 10012, USA

The Avellaneda-Stoikov Framework (Revisited)

One Asset – Price with Brownian Dynamics

One asset with reference price process ("mid"-price) $(S_t)_t$:

$$dS_t = \sigma dW_t.$$

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Market Maker's Bid and Ask Prices

$$S_t^b = S_t - \delta_t^b \text{ and } S_t^a = S_t + \delta_t^a.$$

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Dynamics of the Inventory $(q_t)_t$

$$dq_t = \Delta dN_t^b - \Delta dN_t^a, \qquad \text{for two point processes } N^b \text{ and } N^a.$$

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Competition and demand are modeled indirecty through the probability / intensity of jumps.

The Avellaneda-Stoikov Framework (Revisited)

Intensities $(\lambda_t^b)_t$ and $(\lambda_t^a)_t$ of N^b and N^a

$$\lambda_t^b = \Lambda^b(\delta_t^b) 1_{q_{t-} < Q} \text{ and } \lambda_t^a = \Lambda^a(\delta_t^a) 1_{q_{t-} > -Q}.$$

They depend on the distance to the reference price: Λ^b, Λ^a **decreasing** (of course!)

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Cash Process $(X_t)_t$

$$dX_t = \Delta S_t^a dN_t^a - \Delta S_t^b dN_t^b = -S_t dq_t + \delta_t^a \Delta dN_t^a + \delta_t^b \Delta dN_t^b.$$

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State Variables

The most parsimonious model has 3 state variables: X (cash), q (inventory), and S (price).

PnL and Objective Function

Market Maker's PnL

$$PnL_T = X_T + q_T S_T = X_0 + q_0 S_0$$

$$+ \int_0^T \underbrace{\delta_t^a \Delta dN_t^a + \delta_t^b \Delta dN_t^b}_{\text{spread capture}} + \underbrace{\sigma q_t dW_t}_{\text{inventory+price risk}}$$

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Von Neumann-Morgenstern Criterion

The original Avellaneda-Stoikov's model considers a CARA utility (Model A):

$$\sup_{(\delta_t^a)_t, (\delta_t^b)_t \in \mathcal{A}} \mathbb{E}\left[-\exp\left(-\gamma(X_T + q_T S_T)\right)\right],$$

where γ is a risk aversion parameter, and $\mathcal{A}$ the set of predictable processes bounded from below.

HJB Equation

HJB Equation

In what follows, u is a candidate for the value function.

Hamilton-Jacobi-Bellman

$$
\begin{aligned}
\text{(HJB)} \qquad 0 = \partial_t u(t,x,q,S) + \frac{1}{2}\sigma^2 \partial^2_{SS} u(t,x,q,S) \\
+ 1_{q<Q} \sup_{\delta^b} \Lambda^b(\delta^b)\left[u(t, x - \Delta S + \Delta\delta^b, q + \Delta, S) - u(t,x,q,S)\right] \\
+ 1_{q>-Q} \sup_{\delta^a} \Lambda^a(\delta^a)\left[u(t, x + \Delta S + \Delta\delta^a, q - \Delta, S) - u(t,x,q,S)\right]
\end{aligned}
$$

with final condition:

$$
u(T,x,q,S) = -\exp\left(-\gamma(x + qS)\right)
$$

Change of Variables

Ansatz

$$u(t, x, q, S) = -\exp(-\gamma(x + qS + \theta(t, q)))$$

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New Equation

$$0 = \partial_t \theta(t, q) - \frac{1}{2}\gamma\sigma^2 q^2$$
$$+1_{q<Q} \sup_{\delta^b} \frac{\Lambda^b(\delta^b)}{\gamma} \left(1 - \exp\left(-\gamma\left(\Delta\delta^b + \theta(t, q+\Delta) - \theta(t, q)\right)\right)\right)$$
$$+1_{q>-Q} \sup_{\delta^a} \frac{\Lambda^a(\delta^a)}{\gamma} \left(1 - \exp\left(-\gamma\left(\Delta\delta^a + \theta(t, q-\Delta) - \theta(t, q)\right)\right)\right)$$

with final condition $\theta(T, q) = 0$.

Equation for θ

A New Transform

$$H_\xi^b(p) = \sup_\delta \frac{\Lambda^b(\delta)}{\xi\Delta}\left(1 - \exp\left(-\xi\Delta\left(\delta - p\right)\right)\right)$$

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New Equation

$$0 = \partial_t\theta(t,q) - \frac{1}{2}\gamma\sigma^2 q^2 + 1_{q<Q}\Delta H_\gamma^b\left(\frac{\theta(t,q) - \theta(t,q+\Delta)}{\Delta}\right)$$
$$+1_{q>-Q}\Delta H_\gamma^a\left(\frac{\theta(t,q) - \theta(t,q-\Delta)}{\Delta}\right)$$

with final condition $\theta(T,q) = 0$.

Another Objective Function

Risk-Neutral with Running Penalty (Model B)

Variant (Cartea, Jaimungal *et al.*) with a running penalty:

$$\sup_{(\delta_t^a)_t,(\delta_t^b)_t \in \mathcal{A}} \mathbb{E}\left[X_T + q_T S_T - \frac{\gamma}{2}\sigma^2 \int_0^T q_t^2 dt\right]$$

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i.e.

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Remark

Optimal control in continuous time on a very simple finite graph $\Delta\mathbb{Z} \cap [-Q, Q]$ with edges between neighbours.

Value Function θ (Model B)

Hamilton-Jacobi Equation (Model B)

$$0 = \partial_t \theta(t,q) - \frac{1}{2}\gamma\sigma^2 q^2 + 1_{q<Q}\Delta H_0^b\left(\frac{\theta(t,q)-\theta(t,q+\Delta)}{\Delta}\right)$$
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Same Kind of Transform

$$H_0^b(p) = \sup_\delta \Lambda^b(\delta)(\delta - p)$$
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From Modelling to Analysis

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Light Assumptions for the Intensity Functions

1. $\Lambda^{b/a}$ is C^2.
2. ${\Lambda^{b/a}}' < 0$.
3. $\lim_{\delta \to +\infty} \Lambda^{b/a}(\delta) = 0$.
4. The intensity functions $\Lambda^{b/a}$ satisfy:

$$\sup_{\delta} \frac{\Lambda^{b/a}(\delta){\Lambda^{b/a}}''(\delta)}{\left({\Lambda^{b/a}}'(\delta)\right)^2} < 2.$$

Simple Mathematical Results

Proposition

- $\forall \xi \geq 0$, $H_\xi^{b/a}$ is a decreasing function of class C^2.
- In the definition of $H_\xi^{b/a}(p)$, the supremum is attained at a unique $\tilde{\delta}_\xi^{b/a*}(p)$ characterized by

$$\tilde{\delta}_\xi^{b/a*}(p) = {\Lambda^{b/a}}^{-1}\left(\xi \Delta H_\xi^{b/a}(p) - {H_\xi^{b/a}}'(p)\right).$$

- The function $p \mapsto \tilde{\delta}_\xi^{b/a*}(p)$ is increasing.

Existence and Uniqueness

Theorem

There exists a unique C^1 (in time) solution $t \mapsto (\theta(t,q))_{|q| \le Q}$ to

$$0 = \partial_t \theta(t,q) - \frac{1}{2}\gamma\sigma^2 q^2 + 1_{q<Q} \Delta H^b_\xi \left(\frac{\theta(t,q) - \theta(t,q+\Delta)}{\Delta} \right)$$

$$+ 1_{q>-Q} \Delta H^a_\xi \left(\frac{\theta(t,q) - \theta(t,q-\Delta)}{\Delta} \right)$$

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Ingredients of the Proof

- Cauchy-Lipschitz for local existence and uniqueness (ODE tools).
- Comparison principle (HJ tools).
- Exhibition of locally bounded sub- and super-solutions $\rightarrow$ No blow-up.

Solution of the Initial Control Problem

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Optimal Quotes

The optimal quotes in models A ($\xi = \gamma$) and B ($\xi = 0$) are:

$$\delta_t^{b*} = \tilde{\delta}_\xi^{b*}\left(\frac{\theta(t, q_{t-}) - \theta(t, q_{t-} + \Delta)}{\Delta}\right)$$

$$\delta_t^{a*} = \tilde{\delta}_\xi^{a*}\left(\frac{\theta(t, q_{t-}) - \theta(t, q_{t-} - \Delta)}{\Delta}\right)$$

where

$$\tilde{\delta}_\xi^{b/a*}(p) = {\Lambda^{b/a}}^{-1}\left(\xi \Delta H_\xi^{b/a}(p) - {H_\xi^{b/a}}'(p)\right).$$

Special Case: $\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta}$

Closed-Form Expressions

The Hamiltonian functions and the maximizers are given by

$$H_\xi^{b/a}(p) = \frac{A}{k} C_\xi \exp(-kp), \quad \text{with } C_\xi = \begin{cases} \left(1 + \frac{\xi\Delta}{k}\right)^{-\frac{k}{\xi\Delta}-1} & \text{if } \xi > 0 \\ e^{-1} & \text{if } \xi = 0 \end{cases};$$

$$\tilde{\delta}_\xi^{b/a*}(p) = \begin{cases} p + \frac{1}{\xi\Delta}\log\left(1 + \frac{\xi\Delta}{k}\right) & \text{if } \xi > 0 \\ p + \frac{1}{k} & \text{if } \xi = 0. \end{cases}$$

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The System of ODEs

$$0 = \partial_t \theta(t,q) - \frac{1}{2}\gamma\sigma^2 q^2 + \frac{A\Delta}{k} C_\xi \left(1_{q<Q} e^{k\frac{\theta(t,q+\Delta)-\theta(t,q)}{\Delta}} + 1_{q>-Q} e^{k\frac{\theta(t,q-\Delta)-\theta(t,q)}{\Delta}}\right), \quad \text{and } \theta(T,q) = 0.$$

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Change of variables: $v_q(t) = \exp\left(\frac{k\theta(t,q)}{\Delta}\right)$

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A **Linear** System of ODEs

$$v_q'(t) = \alpha q^2 v_q(t) - \eta_\xi \left(1_{q<Q} v_{q+\Delta}(t) + 1_{q>-Q} v_{q-\Delta}(t)\right), \quad \text{and } v(T,q) = 1$$

with

$$\alpha = \frac{k}{2\Delta}\gamma\sigma^2 \quad \text{and} \quad \eta_\xi = AC_\xi.$$

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We have a **closed-form** solution through matrix exponentiation and we can easily study the **ergodic behaviour**.

Special Case: $\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta}$

Asymptotics

$$\delta_\infty^{b*}(q) = \lim_{T\to\infty} \delta^{b*}(0,q) = D_\xi + \frac{1}{k}\ln\left(\frac{f_q^0}{f_{q+\Delta}^0}\right) \qquad \delta_\infty^{a*}(q) = \lim_{T\to\infty} \delta^{a*}(0,q) = D_\xi + \frac{1}{k}\ln\left(\frac{f_q^0}{f_{q-\Delta}^0}\right)$$

with

$$D_\xi = \begin{cases} \frac{1}{\xi\Delta}\log\left(1+\frac{\xi\Delta}{k}\right) & \text{if } \xi > 0 \\ \frac{1}{k} & \text{if } \xi = 0, \end{cases}$$

Variational Characterization

$f^0 \in \mathbb{R}^{2Q+1}$ is characterized by:

$$\operatorname*{argmin}_{\|f\|_2=1} \sum_{|q|\le Q} \alpha q^2 f_q^2 + \eta_\xi \left(\sum_{q=-Q}^{Q-\Delta} (f_{q+\Delta} - f_q)^2 + (f_Q)^2 + (f_{-Q})^2\right).$$

Special Case: $\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta}$

Continuous Counterpart

$\tilde{f}^0 \in L^2(\mathbb{R})$ characterized by:

$$\underset{\|\tilde{f}\|_{L^2(\mathbb{R})}=1}{\operatorname{argmin}} \int_{-\infty}^{\infty} \left(\alpha x^2 \tilde{f}(x)^2 + \eta_\xi \Delta^2 \tilde{f}'(x)^2\right) dx.$$

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$$\tilde{f}^0(x) \propto \exp\left(-\frac{1}{2\Delta} \sqrt{\frac{\alpha}{\eta_\xi}} x^2 \right)$$

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$$\tilde{f}^0(x) \propto \exp\left(-\frac{1}{2\Delta}\sqrt{\frac{\alpha}{\eta_\xi}}x^2\right)$$

Hence, we get an approximation of the form:

$$f_q^0 \propto \exp\left(-\frac{1}{2\Delta}\sqrt{\frac{\alpha}{\eta_\xi}}q^2\right)$$

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Closed-Form Approximations (Model A: $\xi = \gamma$)

$$
\begin{aligned}
\delta_\infty^{b*}(q) &\simeq \frac{1}{\Delta\xi}\ln\left(1+\frac{\Delta\xi}{k}\right) + \frac{2q+\Delta}{2}\sqrt{\frac{\gamma\sigma^2}{2kA\Delta}\left(1+\frac{\Delta\xi}{k}\right)^{1+\frac{k}{\Delta\xi}}} \\
\delta_\infty^{a*}(q) &\simeq \frac{1}{\Delta\xi}\ln\left(1+\frac{\Delta\xi}{k}\right) - \frac{2q-\Delta}{2}\sqrt{\frac{\gamma\sigma^2}{2kA\Delta}\left(1+\frac{\Delta\xi}{k}\right)^{1+\frac{k}{\Delta\xi}}}
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$$\delta_\infty^{a*}(q) \simeq \frac{1}{\Delta\xi}\ln\left(1+\frac{\Delta\xi}{k}\right) - \frac{2q-\Delta}{2}\sqrt{\frac{\gamma\sigma^2}{2kA\Delta}\left(1+\frac{\Delta\xi}{k}\right)^{1+\frac{k}{\Delta\xi}}}$$

Closed-Form Approximations (Model B: $\xi = 0$)

$$\delta_\infty^{b*}(q) \simeq \frac{1}{k} + \frac{2q+\Delta}{2}\sqrt{\frac{\gamma\sigma^2 e}{2kA\Delta}}$$

$$\delta_\infty^{a*}(q) \simeq \frac{1}{k} - \frac{2q-\Delta}{2}\sqrt{\frac{\gamma\sigma^2 e}{2kA\Delta}}$$

Extensions and Industry Applications

Important Practical Considerations

- Multiple sizes of transactions (one quote per size).
- Multiple tiers of clients (quotes per tier)
- Modeling adverse selection.
- Internalization vs. Externalization.
- **Going multi-asset without suffering from the curse of dimensionality in computations – no-grid methods.**

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Applied by bond and FX dealers (+ AMM)

Time for a short animation

Going Back to Optimal Control Theory

Our work on market making led to broader questions in continuous-time optimal control on graphs.

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Existence and Uniqueness

Classical methods from ODEs along with first-order Hamilton-Jacobi (HJ) tools can be adapted to prove general existence and uniqueness results.

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Results for the long-term behaviour under the assumption of **strictly monotone Hamiltonians**.

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Special Case: Entropic Costs

In the special case of **entropic costs/penalties**, the non-linear Bellman equation can be transformed into a **system of linear ODEs** – exponential/entropy duality linked to our exponential intensities. This allows the ergodic behaviour to be studied directly using **matrix analysis**.

The Role of Entropy in Continuous-Time Optimal Control

1D Problem Formulation

Consider a state process X taking values in $\mathbb{R}$, governed by the SDE:

$$dX_t = a_t dt + \sigma dW_t$$

where a_t is the control and σ the diffusion coefficient.

Objective Function

Maximize the expected reward minus a quadratic effort penalty $\alpha > 0$:

$$\sup_{a \in \mathcal{A}} \mathbb{E}^{\mathbb{P}} \left[\int_0^T \left(f(X_t) - \alpha a_t^2 \right) dt + g(X_T) \right]$$

f and g are assumed concave (e.g., quadratic for the LQG case).

Measure Transformation and the Base Process

The Base Process Y

Define the uncontrolled base process Y as a pure diffusion:

$$dY_t = \sigma dW_t, \quad Y_0 = X_0$$

Girsanov Transformation

For any admissible a, define the shift $\theta_t = a_t/\sigma$. We introduce the measure $\mathbb{P}^a$ via:

$$\frac{d\mathbb{P}^a}{d\mathbb{P}} = \exp\left(\int_0^T \theta_t dW_t - \frac{1}{2}\int_0^T \theta_t^2 dt\right)$$

Under $\mathbb{P}^a$, the dynamics of Y match the controlled dynamics of X:

$$dY_t = a_t dt + \sigma dW_t^a, \quad \text{where } W^a \text{ is a } \mathbb{P}^a\text{-Brownian motion.}$$

Entropy Mapping and Variational Principles (I)

Objective under $\mathbb{P}^a$

The problem is rewritten over the fixed trajectories of Y:

$$\sup_a \mathbb{E}^{\mathbb{P}^a}\left[\int_0^T \left(f(Y_t) - \alpha a_t^2\right) dt + g(Y_T)\right]$$

KL Divergence

The relative entropy between the controlled measure $\mathbb{P}^a$ and base measure $\mathbb{P}$ is:

$$D_{\mathrm{KL}}(\mathbb{P}^a \| \mathbb{P}) = \mathbb{E}^{\mathbb{P}^a}\left[\log \frac{d\mathbb{P}^a}{d\mathbb{P}}\right] = \frac{1}{2}\mathbb{E}^{\mathbb{P}^a}\left[\int_0^T \frac{a_t^2}{\sigma^2} dt\right]$$

By defining $c = 2\sigma^2\alpha$, the penalty becomes $cD_{\mathrm{KL}}(\mathbb{P}^a \| \mathbb{P})$.

Entropy Mapping and Variational Principles (II)

Donsker-Varadhan / Dupuis-Boué

The control problem maps to a variational problem over measures $\mathbb{Q} \ll \mathbb{P}$:

$$\sup_{\mathbb{Q} \ll \mathbb{P}} \left\{ \mathbb{E}^{\mathbb{Q}} \left[\int_0^T f(Y_t) dt + g(Y_T) \right] - c D_{\mathrm{KL}}(\mathbb{Q} \| \mathbb{P}) \right\}$$

The solution is the value function $v(t, x)$ (where the optimal value is $v(0, x_0)$):

$$v(t, x) = c \log \mathbb{E}^{\mathbb{P}} \left[\exp \left(\frac{1}{c} \left(\int_t^T f(Y_s) ds + g(Y_T) \right) \right) \middle| Y_t = x \right]$$

Classes of Solvable Problems

Beyond the LQG Paradigm

While f, g quadratic corresponds to the classical Riccati-based LQG, the Log-Laplace representation allows closed-form solutions for any potential where the previous expectation is known.

Explicit Examples (beyond quadratic)

- **Linear Costs:** $f(x) = \beta x$.
- **Pure Terminal Payoff:** $f = 0$, solved via heat kernel convolution.
- Other potentials from physics (ask Gemini/Claude/ChatGPT... the list is long!)

Optimal Control on Graphs

State Process X_t

Let X_t be a Continuous-Time Markov Chain on a finite graph $\mathcal{X}$. The uncontrolled dynamics are defined by base intensities $(\bar{\lambda}_{ij})_{i,j \in \mathcal{X}}$.

Point Process Girsanov (Brémaud's Formula)

For controlled intensities $\lambda_{ij}(t)$, the Radon-Nikodym derivative is:

$$\log \frac{d\mathbb{P}^{\lambda}}{d\mathbb{P}^{\bar{\lambda}}} = \int_0^T \sum_{j \in \mathcal{V}(X_{s-})} \log \left(\frac{\lambda_{X_{s-},j}(s)}{\bar{\lambda}_{X_{s-},j}} \right) dN_{X_{s-},j}(s) - \int_0^T \sum_{j \neq X_s} (\lambda_{X_s,j}(s) - \bar{\lambda}_{X_s,j}) ds$$

where $N_{ij}(t)$ counts transitions from state i to state j.

Entropic Costs on Graphs

KL Divergence

$$D_{\mathrm{KL}}(\mathbb{P}^{\lambda} \| \mathbb{P}^{\bar{\lambda}}) = \mathbb{E}^{\mathbb{P}^{\lambda}}\left[\int_0^T \sum_{j \in \mathcal{V}(X_{s-})} \underbrace{\left(\lambda_{X_s,j} \log \frac{\lambda_{X_s,j}}{\bar{\lambda}_{X_s,j}} - \lambda_{X_s,j} + \bar{\lambda}_{X_s,j}\right)}_{\ell(X_s,\lambda)} ds\right]$$

Duality and Control Problem

To maximize $\mathbb{E}^{\mathbb{P}^{\lambda}}\left[\int_0^T f(X_t)dt + g(X_T) - c\int_0^T \ell(X_t,\lambda)dt\right]$, the discrete Donsker-Varadhan duality gives the optimal value:

$$v_0 = c \log \mathbb{E}^{\mathbb{P}^{\bar{\lambda}}}\left[\exp\left(\frac{1}{c}\left(\int_0^T f(X_t)dt + g(X_T)\right)\right)\right]$$

The optimal measure $\mathbb{Q}^*$ has density $\frac{d\mathbb{Q}^*}{d\mathbb{P}^{\bar{\lambda}}} \propto \exp(S(X)/c)$.

Discrete Value Function and Optimal Rates

The Value Function $v(t, i)$

The value function is defined as the conditional log-expectation:

$$v(t, i) = c \log \mathbb{E}^{\mathbb{P}^{\bar{\lambda}}} \left[\exp\left(\frac{1}{c} \left(\int_t^T f(X_s) ds + g(X_T) \right) \right) \middle| X_t = i \right]$$

It is linearized via the Cole-Hopf transform $w(t, i) = \exp(v(t, i)/c)$:

$$\frac{d}{dt} w(t, i) + \sum_{j \neq i} \bar{\lambda}_{ij} (w(t, j) - w(t, i)) + \frac{1}{c} f(i) w(t, i) = 0$$

Optimal Intensities

The optimal controlled jump rates satisfy the "tilted" intensities:

$$\lambda^*_{ij}(t) = \bar{\lambda}_{ij} \exp\left(\frac{v(t, j) - v(t, i)}{c} \right)$$

Avellaneda-Stoikov: From Quotes to Intensities

The Market Maker's Objective

The goal is to maximize execution revenue minus inventory risk penalty:

$$\sup_{\delta^b,\delta^a} \mathbb{E}\left[\int_0^T \left(\sum_{z\in\{b,a\}} \lambda_t^z \delta_t^z \Delta - \frac{\gamma\sigma^2}{2} q_t^2\right) dt\right]$$

where the intensities follow the model $\lambda^z = Ae^{-k\delta^z}$.

Avellaneda-Stoikov: From Quotes to Intensities

Algebraic Mapping to Entropy

Inverting the intensity gives $\delta = -\frac{1}{k}\log(\frac{\lambda}{A})$. The revenue rate is:

$$\lambda\delta\Delta = -\frac{\Delta}{k}\lambda\log\left(\frac{\lambda}{A}\right)$$

With $\bar{\lambda} = Ae^{-1}$, we use $\lambda\log(\frac{\lambda}{A}) = \ell(\lambda, \bar{\lambda}) - \bar{\lambda}$ to obtain:

$$\lambda\delta\Delta = -\frac{\Delta}{k}\ell(\lambda, \bar{\lambda}) + \frac{\Delta}{k}\bar{\lambda}$$

where $\ell(\lambda, \bar{\lambda}) = \lambda\log(\lambda/\bar{\lambda}) - \lambda + \bar{\lambda}$ is the entropy rate.

Avellaneda-Stoikov: The Entropic Solution

The Isomorphic Control Problem

The problem maps exactly to an entropic-form optimal control problem on the graph $\Delta\mathbb{Z}$:

$$\sup_{\lambda^b,\lambda^a} \mathbb{E}^{\mathbb{P}^\lambda}\left[\int_0^T \left(f(q_t) - c \sum_{z\in\{b,a\}} \ell(\lambda_t^z, \bar{\lambda})\right) dt\right] + 2c\bar{\lambda}T$$

with the following parameter identifications:

- $f(q) = -\frac{\gamma\sigma^2}{2}q^2$ (Running inventory penalty)
- $c = \frac{\Delta}{k}$ (Scaling parameter)
- $\bar{\lambda} = Ae^{-1}$ (Base jump intensity for bid and ask)

Questions?