Clustered Network Connectedness: A New Measurement Framework with Application to Global Equity Markets

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Types of connectedness (Forbes and Rigobon, 2002)

- **Co-movement**: captures interdependence of assets without being able to pinpoint the origin of the shock
- **Contagion**: captures how shocks occurring to one asset impact shocks in other assets (requires identification strategy)

Measuring connectedness

- **Impulse response functions** (IRFs): the evolution of the variable of interest along a specified time horizon after a shock in a given moment
- **Forecast Error Variance Decompositions** (VDs): quantify how much of the unforeseen variation is to the shocks occurring to the asset it self vs. shocks occurring to other assets

Contagion: orthogonalized IRFs (Sims, 1980)

- Diebold and Yilmaz (2009), Giglio et al. (2016), Barunik and Krehlik (2018)
- Swanson and Granger (1997), Klößner and Wagner (2014)

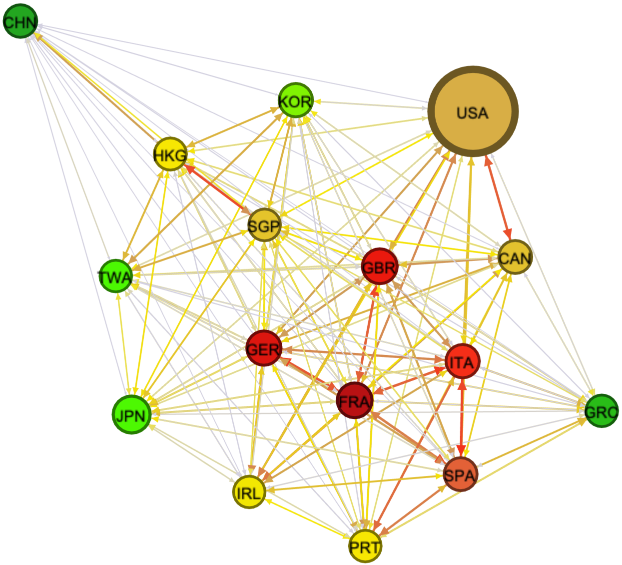
Co-movements: generalized IRFs (Koop et al.,1996)

- Diebold and Yilmaz (2012), Bekaert, et al. (2014), Basak and Pavlova (2016) Corbet et al. (2018)

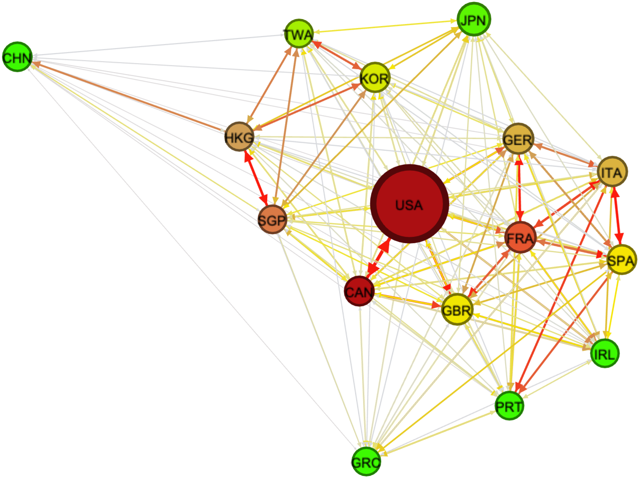
This paper:

- Develops a unified approach which contains orthogonalized IRFs and generalized IRFs as corner solutions
- Provides are framework that includes causal connectedness across clusters while capturing heterogenous dynamics within clusters.
- Application to global stock markets: 16 countries spanning 3 regions

Preview of results: Generalized approach



Preview of results: Clustered approach



1. Measuring Network Connectedness
   - Vector Autoregression Framework
   - Orthogonalized Impulse Response Functions
   - Generalized Impulse Response Functions

2. Clustering
   - Cluster-Orthogonalized Impulse Response Functions
   - Cluster-Orthogonalized Variance Decompositions
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3. Empirical Implementation
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4. Results
   - Full Sample
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– Part 1 –

Measuring Network Connectedness

- Vector Autoregression Framework
- Orthogonalized Impulse Response Functions
- Generalized Impulse Response Functions

Consider a covariance stationary N-variable vector autoregression (VAR) with P lags:

$$\boldsymbol{x}_t = \sum_{p=1}^{P} \boldsymbol{\Phi}_p \boldsymbol{x}_{t-p} + \boldsymbol{u}_t = \sum_{i=0}^{\infty} \boldsymbol{A}_i \boldsymbol{u}_{t-i}, \qquad t = 1, 2, \ldots, T \tag{1}$$

where

- $\mathbb{E}[\boldsymbol{u}_t] = 0$
- $\mathbb{V}[\boldsymbol{u}_t] = \boldsymbol{\Sigma}$ with $\boldsymbol{\Sigma} = \{\sigma_{ij}, i, j = 1, 2, \ldots N\}$.
- $\boldsymbol{x}_t = [\ x_{1t}\ x_{2t}\ \ldots\ x_{Nt}\ ]^\top$ is a $[N \times 1]$ vector of asset returns

Other specifications

- Φ_p is a $N \times N$ parameter matrix for lag p.
- A_0 is an identity matrix of size N and $A_i = \Phi_1 A_{i-1} + \Phi_2 A_{i-2} + \cdots + \Phi_p A_{i-p}$ with $A_i = 0$ for $i < 0$.

For any lower-triangular non-singular $[N \times N]$ matrix $\boldsymbol{Q}_C$, we can rewrite the moving average representation in (1) without loss of generality as

$$\boldsymbol{x}_t = \sum_{i=0}^{\infty} \boldsymbol{A}_i \boldsymbol{u}_{t-i} = \sum_{i=0}^{\infty} \boldsymbol{A}_i \boldsymbol{Q}_C \boldsymbol{Q}_C^{-1} \boldsymbol{u}_{t-i} = \sum_{i=0}^{\infty} \boldsymbol{A}_i \boldsymbol{Q}_C \boldsymbol{\epsilon}_{t-i}$$

where

- $\mathbb{E}[\boldsymbol{\epsilon}_t] = \mathbb{E}[\boldsymbol{Q}_C^{-1} \boldsymbol{u}_t] = 0$.
- $\mathbb{V}[\boldsymbol{\epsilon}_t] = \mathbb{V}[\boldsymbol{Q}_C^{-1} \boldsymbol{u}_t] = \boldsymbol{\Omega}_C$ with $\boldsymbol{\Omega}_C = \{\omega_{C,ij}, i, j = 1, 2, \ldots N\}$.

Following Koop et al. (1996) the impulse response function from a shock of the j^{th} asset on $\boldsymbol{x}_{t+h}$ is given by

$$\boldsymbol{\psi}_j^C(h) = \frac{\boldsymbol{A}_h \boldsymbol{Q}_C \boldsymbol{\Omega}_C \boldsymbol{e}_j}{\sqrt{\omega_{C,jj}}} \tag{2}$$

where $\boldsymbol{e}_j$ is a $[N \times 1]$ selection vector which is equal to one in the j^{th} position, and zero elsewhere.

More

Matrix $\boldsymbol{Q}_C$

- C will denote the number of clusters,

$\rightarrow$ it determines the structure of $\boldsymbol{Q}_C$ and hence $\boldsymbol{\Omega}_C$

Any number of clusters $C \in [1, N]$ could be operative!

The literature has so far only focused on two very special cases:

- $C = N$, where each network node is its own cluster (*orthogonalized IRFs*)
- $C = 1$, where all nodes are grouped into a single cluster (*generalized IRFs*)

Approach of Sims (1980): uncorrelated residuals

- Mathematically, $C = N$, as many clusters, C, as assets, N.
- $\boldsymbol{Q}_N = \boldsymbol{M}$ where $\boldsymbol{M}$ is the unique lower triangular matrix from the Cholesky decomposition of $\boldsymbol{\Sigma} = \boldsymbol{M}\boldsymbol{M}^\top$.
- $\boldsymbol{\Omega}_N = \mathbb{V}[\boldsymbol{Q}_N^{-1}\boldsymbol{u}_{t-i}] = \mathbb{V}[\boldsymbol{M}^{-1}\boldsymbol{u}_{t-i}] = \boldsymbol{I}_N$, which means $\sqrt{\omega_{N,jj}} = 1$,

Orthogonalized impulse response functions

$$\boldsymbol{\psi}_j^N(h) = \frac{\boldsymbol{A}_h\boldsymbol{Q}_N\boldsymbol{\Omega}_N\boldsymbol{e}_j}{\sqrt{\omega_{N,jj}}} = \frac{\boldsymbol{A}_h\boldsymbol{M}\boldsymbol{I}_N\boldsymbol{e}_j}{\sqrt{1}} = \boldsymbol{A}_h\boldsymbol{M}\boldsymbol{e}_j = \boldsymbol{\psi}_j^o(h) \tag{3}$$

Note:
It is paramount to notice that the structure of the lower triangular Cholesky matrix crucially depends on the order of the variables in the VAR (Swanson and Granger, 1997).

$$
\boldsymbol{\Omega}_N = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & \dots & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & \dots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1
\end{pmatrix} = \begin{pmatrix}
\mathbf{I} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{I} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{I}
\end{pmatrix}
$$

Pros and Cons

+ Causal interpretation
- Need to select a single identification strategy out of $N!$ possible ones
- Not able to accommodate similarities within clusters

Approach of Koop, Pesaran and Potter (1996): correlated residuals

- Mathematically, $C = 1$, i.e all N nodes belong to the same cluster
- $\boldsymbol{Q}_1 = \boldsymbol{I}_N$
- $\boldsymbol{\Omega}_1 = \mathbb{V}[\boldsymbol{Q}_1^{-1}\boldsymbol{u}_t] = \mathbb{V}[\boldsymbol{I}_N\boldsymbol{u}_t] = \mathbb{V}[\boldsymbol{u}_t] = \boldsymbol{\Sigma}$, which means $\sqrt{\omega_{1,jj}} = \sigma_{jj}$,

Generalized Impulse Response Functions

$$\boldsymbol{\psi}_j^1(h) = \frac{\boldsymbol{A}_h\boldsymbol{Q}_1\boldsymbol{\Omega}_1\boldsymbol{e}_j}{\sqrt{\omega_{1,jj}}} = \frac{\boldsymbol{A}_h\boldsymbol{I}_N\boldsymbol{\Sigma}\boldsymbol{e}_j}{\sqrt{\omega_{1,jj}}} = \frac{\boldsymbol{A}_h\boldsymbol{\Sigma}\boldsymbol{e}_j}{\sqrt{\sigma_{jj}}} = \boldsymbol{\psi}_j^g(h) \tag{4}$$

$$
\boldsymbol{\Omega}_1 = \begin{pmatrix}
\boxed{\begin{matrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{matrix}} & \begin{matrix} \sigma_{14} & \sigma_{15} \\ \sigma_{24} & \sigma_{25} \\ \sigma_{34} & \sigma_{35} \end{matrix} & \begin{matrix} \sigma_{16} & \sigma_{17} & \cdots & \sigma_{1N} \\ \sigma_{26} & \sigma_{27} & \cdots & \sigma_{2N} \\ \sigma_{36} & \sigma_{37} & \cdots & \sigma_{3N} \end{matrix} \\
\begin{matrix} \sigma_{41} & \sigma_{42} & \sigma_{43} \\ \sigma_{51} & \sigma_{52} & \sigma_{53} \end{matrix} & \boxed{\begin{matrix} \sigma_{44} & \sigma_{45} \\ \sigma_{54} & \sigma_{55} \end{matrix}} & \begin{matrix} \sigma_{46} & \sigma_{47} & \cdots & \sigma_{4N} \\ \sigma_{56} & \sigma_{57} & \cdots & \sigma_{5N} \end{matrix} \\
\begin{matrix} \sigma_{61} & \sigma_{62} & \sigma_{63} \\ \sigma_{71} & \sigma_{72} & \sigma_{73} \\ \vdots & \vdots & \vdots \\ \sigma_{N1} & \sigma_{N2} & \sigma_{N3} \end{matrix} & \begin{matrix} \sigma_{64} & \sigma_{65} \\ \sigma_{74} & \sigma_{75} \\ \vdots & \vdots \\ \sigma_{N4} & \sigma_{N5} \end{matrix} & \boxed{\begin{matrix} \sigma_{66} & \sigma_{67} & \cdots & \sigma_{6N} \\ \sigma_{76} & \sigma_{77} & \cdots & \sigma_{7N} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{N6} & \sigma_{N7} & \cdots & \sigma_{NN} \end{matrix}}
\end{pmatrix}
= \begin{pmatrix}
\boxed{\boldsymbol{\Sigma}_{11}} & \boldsymbol{\Sigma}_{12} & \boldsymbol{\Sigma}_{13} \\
\boldsymbol{\Sigma}_{21} & \boxed{\boldsymbol{\Sigma}_{22}} & \boldsymbol{\Sigma}_{23} \\
\boldsymbol{\Sigma}_{31} & \boldsymbol{\Sigma}_{32} & \boxed{\boldsymbol{\Sigma}_{33}}
\end{pmatrix}
$$

Pros and Cons

+ No need for identification strategy
- No causal interpretation
- Not able to accommodate similarities within clusters

Goal: manage the trade-off

- orthogonalized IRFs (requires identification strategies) $\rightarrow$ Contagion
- generalized IRFs (no causal interpretation) $\rightarrow$ Co-movement

This paper: find $\boldsymbol{Q}_C$ such that

- residuals are orthogonalized across clusters (causality)
- residuals are correlated within clusters (capture rich dynamics)

$\Rightarrow$ Large number of assets, few identifications strategies

– Part 2 –

Clustering

- Cluster-Orthogonalized Impulse Responses
- Cluster-Orthogonalized Variance Decompositions
- Connectedness Measurement Within and Across Clusters

$$\begin{pmatrix}
\sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} & \sigma_{15} & \sigma_{16} & \sigma_{17} & \cdots & \sigma_{1N} \\
\sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} & \sigma_{25} & \sigma_{26} & \sigma_{27} & \cdots & \sigma_{2N} \\
\sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{34} & \sigma_{35} & \sigma_{36} & \sigma_{37} & \cdots & \sigma_{3N} \\
\sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44} & \sigma_{45} & \sigma_{46} & \sigma_{47} & \cdots & \sigma_{4N} \\
\sigma_{51} & \sigma_{52} & \sigma_{53} & \sigma_{54} & \sigma_{55} & \sigma_{56} & \sigma_{57} & \cdots & \sigma_{5N} \\
\sigma_{61} & \sigma_{62} & \sigma_{63} & \sigma_{64} & \sigma_{65} & \sigma_{66} & \sigma_{67} & \cdots & \sigma_{6N} \\
\sigma_{71} & \sigma_{72} & \sigma_{73} & \sigma_{74} & \sigma_{75} & \sigma_{76} & \sigma_{77} & \cdots & \sigma_{7N} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\sigma_{N1} & \sigma_{N2} & \sigma_{N3} & \sigma_{N4} & \sigma_{N5} & \sigma_{N6} & \sigma_{N7} & \cdots & \sigma_{NN}
\end{pmatrix}
=
\begin{pmatrix}
\mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} & \mathbf{\Sigma}_{13} \\
\mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{22} & \mathbf{\Sigma}_{23} \\
\mathbf{\Sigma}_{31} & \mathbf{\Sigma}_{32} & \mathbf{\Sigma}_{33}
\end{pmatrix}$$

$$
\begin{pmatrix}
\sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} & \sigma_{15} & 0 & 0 & \cdots & 0 \\
\sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} & \sigma_{25} & 0 & 0 & \cdots & 0 \\
\sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{34} & \sigma_{35} & 0 & 0 & \cdots & 0 \\
\sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44} & \sigma_{45} & 0 & 0 & \cdots & 0 \\
\sigma_{51} & \sigma_{52} & \sigma_{53} & \sigma_{54} & \sigma_{55} & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & \omega_{3,66} & \omega_{3,76} & \cdots & \omega_{3,N6} \\
0 & 0 & 0 & 0 & 0 & \omega_{3,76} & \omega_{3,77} & \cdots & \omega_{3,N7} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & \omega_{3,N6} & \sigma_{3,N7} & \cdots & \omega_{3,NN}
\end{pmatrix}
=
\begin{pmatrix}
\boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} & \mathbf{0} \\
\boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \boldsymbol{\Omega}_{33}
\end{pmatrix}
$$

Orthogonalization of Residuals (Illustration with C=3)

$$
\begin{pmatrix}
\sigma_{11} & \sigma_{12} & \sigma_{13} & 0 & 0 & 0 & 0 & \dots & 0 \\
\sigma_{21} & \sigma_{22} & \sigma_{23} & 0 & 0 & 0 & 0 & \dots & 0 \\
\sigma_{31} & \sigma_{32} & \sigma_{33} & 0 & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & 0 & \omega_{3,44} & \omega_{3,45} & 0 & 0 & \dots & 0 \\
0 & 0 & 0 & \omega_{3,54} & \omega_{3,55} & 0 & 0 & \dots & 0 \\
0 & 0 & 0 & 0 & 0 & \omega_{3,66} & \omega_{3,76} & \cdots & \omega_{3,N6} \\
0 & 0 & 0 & 0 & 0 & \omega_{3,76} & \omega_{3,77} & \cdots & \omega_{3,N7} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & \omega_{3,N6} & \sigma_{3,N7} & \cdots & \omega_{3,NN}
\end{pmatrix}
=
\begin{pmatrix}
\mathbf{\Sigma_{11}} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{\Omega_{22}} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{\Sigma_{33}}
\end{pmatrix}
$$

Residuals

→ Correlated within clusters

→ Uncorrelated across clusters

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We orthogonalize residuals across market clusters with linear projections (Gram-Schmidt procedure)

$$
\begin{aligned}
\boldsymbol{\epsilon}_{3,t} =& \boldsymbol{u}_{3,t} - \begin{pmatrix}\boldsymbol{\Sigma}_{31} & \boldsymbol{\Sigma}_{32}\end{pmatrix} \begin{pmatrix}\boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22}\end{pmatrix}^{-1} \begin{pmatrix}\boldsymbol{u}_{1,t} \\ \boldsymbol{u}_{2,t}\end{pmatrix} \\
=& \boldsymbol{u}_{3,t} - \begin{pmatrix}\boldsymbol{\Sigma}_{31} & \boldsymbol{\Sigma}_{32}\end{pmatrix} \begin{pmatrix}\boldsymbol{\Sigma}^{11} & \boldsymbol{\Sigma}^{12} \\ \boldsymbol{\Sigma}^{21} & \boldsymbol{\Sigma}^{22}\end{pmatrix} \begin{pmatrix}\boldsymbol{u}_{1t} \\ \boldsymbol{u}_{2,t}\end{pmatrix} \qquad (5) \\
=& \boldsymbol{u}_{3,t} - \begin{pmatrix}\boldsymbol{\Sigma}_{31}\boldsymbol{\Sigma}^{11} + \boldsymbol{\Sigma}_{32}\boldsymbol{\Sigma}^{21} & \boldsymbol{\Sigma}_{31}\boldsymbol{\Sigma}^{12} + \boldsymbol{\Sigma}_{32}\boldsymbol{\Sigma}^{22}\end{pmatrix} \begin{pmatrix}\boldsymbol{u}_{1,t} \\ \boldsymbol{u}_{2,t}\end{pmatrix} \qquad (6) \\
\boldsymbol{\epsilon}_{2,t} =& \boldsymbol{u}_{2,t} - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{u}_{1,t} \qquad (7) \\
\boldsymbol{\epsilon}_{1,t} =& \boldsymbol{u}_{1,t} \qquad (8)
\end{aligned}
$$

where $\boldsymbol{\epsilon}_{1,t}$, $\boldsymbol{\epsilon}_{2,t}$ and $\boldsymbol{\epsilon}_{3,t}$ denote the orthogonalized counterparts of $\boldsymbol{u}_{1,t}$, $\boldsymbol{u}_{2,t}$ and $\boldsymbol{u}_{3,t}$, respectively. Solving for block inverse matrix depicted in equation (5) yields

$$
\begin{pmatrix}\boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22}\end{pmatrix}^{-1} = \begin{pmatrix}\boldsymbol{\Sigma}^{11} & \boldsymbol{\Sigma}^{12} \\ \boldsymbol{\Sigma}^{21} & \boldsymbol{\Sigma}^{22}\end{pmatrix} = \begin{pmatrix}\boldsymbol{\Sigma}_{11}^{-1} + \boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}K\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1} & -\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}K \\ K\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1} & K\end{pmatrix}
$$

where $K = (\boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12})^{-1}$ is referred to as the Schur complement.

Rewriting equations (6), (7) and (8) in matrix notation yields

$$\begin{pmatrix}\boldsymbol{\epsilon}_{1,t}\\ \boldsymbol{\epsilon}_{2,t}\\ \boldsymbol{\epsilon}_{3,t}\end{pmatrix} = \begin{pmatrix}\boldsymbol{u}_{1,t}\\ \boldsymbol{u}_{2,t}\\ \boldsymbol{u}_{3,t}\end{pmatrix} - \begin{pmatrix} 0 & 0 & 0\\ \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1} & 0 & 0\\ \boldsymbol{\Sigma}_{31}\boldsymbol{\Sigma}^{11} + \boldsymbol{\Sigma}_{32}\boldsymbol{\Sigma}^{21} & \boldsymbol{\Sigma}_{31}\boldsymbol{\Sigma}^{12} + \boldsymbol{\Sigma}_{32}\boldsymbol{\Sigma}^{22} & 0\end{pmatrix}\begin{pmatrix}\boldsymbol{u}_{1,t}\\ \boldsymbol{u}_{2,t}\\ \boldsymbol{u}_{3,t}\end{pmatrix}$$

$$= \begin{pmatrix} \boldsymbol{I} & 0 & 0\\ -\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1} & \boldsymbol{I} & 0\\ -\boldsymbol{\Sigma}_{31}\boldsymbol{\Sigma}^{11} - \boldsymbol{\Sigma}_{32}\boldsymbol{\Sigma}^{21} & -\boldsymbol{\Sigma}_{31}\boldsymbol{\Sigma}^{12} - \boldsymbol{\Sigma}_{32}\boldsymbol{\Sigma}^{22} & \boldsymbol{I}\end{pmatrix}\begin{pmatrix}\boldsymbol{u}_{1,t}\\ \boldsymbol{u}_{2,t}\\ \boldsymbol{u}_{3,t}\end{pmatrix}$$

$$\boldsymbol{\epsilon}_t = \boldsymbol{Q}_3^{-1}\boldsymbol{u}_t$$

and

$$\mathbb{V}[\boldsymbol{\epsilon}_t] = \mathbb{V}[\boldsymbol{Q}_3^{-1}\boldsymbol{u}_t] = \boldsymbol{\Omega}_3$$

$$
\boldsymbol{\Omega}_3 = \begin{pmatrix}
\omega_{3,11} & \omega_{3,12} & \omega_{3,13} & 0 & 0 & 0 & 0 & \dots & 0 \\
\omega_{3,21} & \omega_{3,22} & \omega_{3,23} & 0 & 0 & 0 & 0 & \dots & 0 \\
\omega_{3,31} & \omega_{3,32} & \omega_{3,33} & 0 & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & 0 & \omega_{3,44} & \sigma_{3,45} & 0 & 0 & \dots & 0 \\
0 & 0 & 0 & \omega_{3,54} & \sigma_{3,55} & 0 & 0 & \dots & 0 \\
0 & 0 & 0 & 0 & 0 & \omega_{3,66} & \omega_{3,76} & \cdots & \omega_{3,N6} \\
0 & 0 & 0 & 0 & 0 & \omega_{3,76} & \omega_{3,77} & \cdots & \omega_{3,N7} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & \omega_{3,N6} & \sigma_{3,N7} & \cdots & \omega_{3,NN}
\end{pmatrix} = \begin{pmatrix}
\boldsymbol{\Omega}_{3,11} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \boldsymbol{\Omega}_{3,22} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \boldsymbol{\Omega}_{3,33}
\end{pmatrix}
$$

The Cluster-Orthogonalized Impulse Response Functions are then calculated as follows

$$\boldsymbol{\psi}_j^C(h) = \frac{\boldsymbol{A}_h \boldsymbol{Q}_C \boldsymbol{\Omega}_C \boldsymbol{e}_j}{\sqrt{\omega_{C,jj}}}$$

where

- $\boldsymbol{A}_h$ stems from the moving average representation of the VAR
- $\boldsymbol{Q}_C$ captures the order of orthogonalization of residuals across C clusters
- $\boldsymbol{\Omega}_C = \mathbb{V}[\boldsymbol{\epsilon}_t] = \mathbb{V}[\boldsymbol{Q}_C^{-1}\boldsymbol{u}_t]$ variance-covariance matrix of structured residuals
- $\omega_{C,ij}$ with $i, j = 1, 2, \ldots N$ denote all elements of $\boldsymbol{\Omega}_C$
- $\boldsymbol{e}_j$ is a $[N \times 1]$ selection vector which is equal to one in the j^{th} position, and zero elsewhere.

Orthogonalized IRFs (Sims 1980): $C = N$

$\rightarrow$ $\boldsymbol{Q}_N = \boldsymbol{M}$ and $\boldsymbol{\Omega}_N = \boldsymbol{I}_N$

$$\boldsymbol{\psi}_j^N(h) = \boldsymbol{\psi}_j^o(h) = \boldsymbol{A}_h \boldsymbol{M} \boldsymbol{e}_j$$

Clustered IRFs (this paper): $C \in [1, N]$

$$\boldsymbol{\psi}_j^C(h) = \frac{\boldsymbol{A}_h \boldsymbol{Q}_C \boldsymbol{\Omega}_C \boldsymbol{e}_j}{\sqrt{\omega_{C,jj}}}$$

Generalized IRFs (Koop et al. 1996): $C = 1$

$\rightarrow$ $\boldsymbol{Q}_1 = \boldsymbol{I}_N$ and $\boldsymbol{\Omega}_1 = \boldsymbol{\Sigma}$

$$\boldsymbol{\psi}_j^1(h) = \boldsymbol{\psi}_j^g(h) = \frac{\boldsymbol{A}_h \boldsymbol{\Sigma} \boldsymbol{e}_j}{\sqrt{\sigma_{jj}}}$$

Approach:

- Thus far we have focused exclusively on IRFs
- where basic issues and identification concepts are most easily introduced
- VDs (which are simple transformations of IRFs) turn out to be more appealing for constructing and applying actual connectedness measures

Reasons for VD

- VDs (like IRFs) quantify connectedness at the most granular pairwise level: "How much of the H-step-ahead uncertainty in asset return i is due to shocks originating from return j?"
- VDs allow for levels of cross-sectional aggregation beyond pairwise: "How much of the future uncertainty in one return is due to shocks from *all other* returns?"
- VDs also allow for temporal aggregation, via different connectedness strengths at different horizons H, facilitating examination of a variety of horizons

We denote the H-step-ahead VD by $\tilde{\theta}_{ij}^{C}(H)$:

$$\tilde{\theta}_{ij}^{C}(H) = \frac{\sum_{h=0}^{H-1}(\boldsymbol{e}_i'\boldsymbol{\psi}_h)^2}{\sum_{h=0}^{H-1}(\boldsymbol{e}_i'\boldsymbol{A}_h\boldsymbol{\Sigma}\boldsymbol{A}_h'\boldsymbol{e}_i)^2} = \frac{\omega_{C,jj}^{-1}\sum_{h=0}^{H-1}(\boldsymbol{e}_i'\boldsymbol{A}_h\boldsymbol{Q}_C\boldsymbol{\Omega}_C\boldsymbol{e}_j)^2}{\sum_{h=0}^{H-1}(\boldsymbol{e}_i'\boldsymbol{A}_h\boldsymbol{\Sigma}\boldsymbol{A}_h'\boldsymbol{e}_i)^2} \tag{9}$$

where $\tilde{\theta}_{ij}^{C}$ is the share of the H-step-ahead forecast error variance of asset i due to shocks from asset j.

Due to the non-zero covariance of residuals in we note that $\sum_{j=1}^{N}\tilde{\theta}_{ij}^{C}(H) \neq 1$. In line with Diebold and Yilmaz (2012), we normalize to produce $\theta_{ij}^{C}(H) = \frac{\tilde{\theta}_{ij}^{C}(H)}{\sum_{j=1}^{N}\tilde{\theta}_{ij}^{C}(H)}$.

More

$$
\Theta^C(H) = \begin{pmatrix}
\theta_{11} & \theta_{12} & \theta_{13} & \theta_{14} & \theta_{15} & \theta_{16} & \theta_{17} & \cdots & \theta_{1N} \\
\theta_{21} & \theta_{22} & \theta_{23} & \theta_{24} & \theta_{25} & \theta_{26} & \theta_{27} & \cdots & \theta_{2N} \\
\theta_{31} & \theta_{32} & \theta_{33} & \theta_{34} & \theta_{35} & \theta_{36} & \theta_{37} & \cdots & \theta_{3N} \\
\theta_{41} & \theta_{42} & \theta_{43} & \theta_{44} & \theta_{45} & \theta_{46} & \theta_{47} & \cdots & \theta_{4N} \\
\theta_{51} & \theta_{52} & \theta_{53} & \theta_{54} & \theta_{55} & \theta_{56} & \theta_{57} & \cdots & \theta_{5N} \\
\theta_{61} & \theta_{62} & \theta_{63} & \theta_{64} & \theta_{65} & \theta_{66} & \theta_{67} & \cdots & \theta_{6N} \\
\theta_{71} & \theta_{72} & \theta_{73} & \theta_{74} & \theta_{75} & \theta_{76} & \theta_{77} & \cdots & \theta_{7N} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\theta_{N1} & \theta_{N2} & \theta_{N3} & \theta_{N4} & \theta_{N5} & \theta_{N6} & \theta_{N7} & \cdots & \theta_{NN}
\end{pmatrix}
$$

own shocks | co-movement | contagion

Interpretation of spillovers:

- Contagion across clusters
- Co-movement within clusters

Diebold and Yilmaz (2014):

- The matrix of VDs can be viewed as the adjacency matrix of a weighted directed network
- Powerful network perspectives and tools in touch with connectedness measurement

Idea:

- Row (from others)
- Columns (to others)

– Part 3 –

Empirical Implementation

- Data
- Elastic Net Estimation
- Identification Strategy

Three clusters, spanning 16 countries

- North America (2 countries)
- Europe (8 countries)
- East Asia (6 countries)

Data:

- Data source: WRDS
- Sample period: July, 10^{th} 2002 - December, 29^{th} 2021
- Daily nominal local-currency stock market indexes $\rightarrow$ weekly returns

Data: Summary Statistics

Region	Country	Label	Mean	Std	Info	Skew	Kurt
North America	All		6.96	15.60	0.45	-1.42	12.74
North America	United States	USA	8.25	16.65	0.50	-1.02	10.18
North America	Canada	CAN	5.68	16.09	0.35	-1.52	15.14
Europe	All		1.17	18.44	0.06	-0.98	9.07
Europe	France	FRA	4.91	19.75	0.25	-0.69	10.24
Europe	Germany	GER	4.43	19.41	0.23	-1.00	9.97
Europe	United Kingdom	GBR	2.85	16.47	0.17	-0.79	9.40
Europe	Portugal	PRT	-0.36	19.60	-0.02	-1.03	8.66
Europe	Ireland	IRL	3.76	23.79	0.16	-0.95	10.86
Europe	Italy	ITA	0.39	21.18	0.02	-0.76	7.71
Europe	Greece	GRC	-8.32	31.06	-0.27	-0.66	8.89
Europe	Spain	SPA	1.69	20.56	0.08	-0.47	6.70
East Asia	All		4.79	15.52	0.31	-0.83	8.97
East Asia	Japan	JPN	3.74	19.78	0.19	-0.70	8.13
East Asia	China	CHN	5.04	24.59	0.20	-0.69	8.08
East Asia	South Korea	KOR	6.71	20.48	0.33	-0.60	10.02
East Asia	Taiwan	TWA	5.61	19.05	0.29	-0.58	7.49
East Asia	Hong Kong	HKG	4.97	20.52	0.24	-0.46	7.12
East Asia	Singapore	SGP	2.66	15.85	0.17	-0.76	10.87
Global	All		3.25	15.60	0.21	-1.16	10.06

	United States	Canada	France	Germany	United Kingdom	Portugal	Ireland	Italy	Greece	Spain	Japan	China	South Korea	Taiwan	Hong Kong	Singapore
United States	1	0.817	0.7819	0.7672	0.7829	0.5784	0.6582	0.711	0.4496	0.697	0.5496	0.2443	0.5266	0.5302	0.5701	0.6058
Canada	0.817	1	0.7492	0.7274	0.757	0.6009	0.6215	0.6867	0.4682	0.6639	0.5507	0.2415	0.5206	0.5293	0.5599	0.6066
France	0.7819	0.7492	1	0.9281	0.9069	0.7311	0.7648	0.9006	0.5308	0.876	0.6301	0.2328	0.5918	0.5644	0.5938	0.641
Germany	0.7672	0.7274	0.9281	1	0.8532	0.7125	0.7515	0.8609	0.5238	0.8419	0.6335	0.2474	0.6416	0.5962	0.6097	0.6663
United Kingdom	0.7829	0.757	0.9069	0.8532	1	0.6932	0.7654	0.8291	0.5135	0.8074	0.6067	0.2518	0.5919	0.5697	0.6209	0.6636
Portugal	0.5784	0.6009	0.7311	0.7125	0.6932	1	0.6024	0.7644	0.5488	0.7511	0.5268	0.2113	0.4999	0.4765	0.5224	0.5473
Ireland	0.6582	0.6215	0.7648	0.7515	0.7654	0.6024	1	0.7157	0.4428	0.6844	0.534	0.2168	0.5337	0.5151	0.5153	0.5713
Italy	0.711	0.6867	0.9006	0.8609	0.8291	0.7644	0.7157	1	0.562	0.892	0.6025	0.2194	0.5456	0.5162	0.5515	0.612
Greece	0.4496	0.4682	0.5308	0.5238	0.5135	0.5488	0.4428	0.562	1	0.5579	0.4157	0.1834	0.4028	0.3988	0.4054	0.4546
Spain	0.697	0.6639	0.876	0.8419	0.8074	0.7511	0.6844	0.892	0.5579	1	0.5738	0.1999	0.5523	0.5321	0.5533	0.604
Japan	0.5496	0.5507	0.6301	0.6335	0.6067	0.5268	0.534	0.6025	0.4157	0.5738	1	0.2485	0.593	0.5457	0.5894	0.6281
China	0.2443	0.2415	0.2328	0.2474	0.2518	0.2113	0.2168	0.2194	0.1834	0.1999	0.2485	1	0.2783	0.2967	0.4127	0.3344
South Korea	0.5266	0.5206	0.5918	0.6416	0.5919	0.4999	0.5337	0.5456	0.4028	0.5523	0.593	0.2783	1	0.6891	0.6861	0.6901
Taiwan	0.5302	0.5293	0.5644	0.5962	0.5697	0.4765	0.5151	0.5162	0.3988	0.5321	0.5457	0.2967	0.6891	1	0.6602	0.6641
Hong Kong	0.5701	0.5599	0.5938	0.6097	0.6209	0.5224	0.5153	0.5515	0.4054	0.5533	0.5894	0.4127	0.6861	0.6602	1	0.7749
Singapore	0.6058	0.6066	0.641	0.6663	0.6636	0.5473	0.5713	0.612	0.4546	0.604	0.6281	0.3344	0.6901	0.6641	0.7749	1

Equation-by-equation estimation of a 16-variable VAR(3) using adaptive elastic nets (Zou and Zhang, 2009)

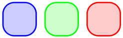
- Ridge regression: squared penalty
- LASSO regression: absolute penalty

In particular, for each equation we solve

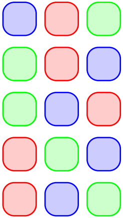
$$\widehat{\boldsymbol{\beta}} = \operatorname{argmin}_{\boldsymbol{\beta}} \left(\sum_{t=1}^{T} \left(y_t - \sum_i \beta_i x_{it} \right)^2 + \lambda \sum_{i=1}^{K} w_i \left(\frac{1}{2} |\beta_i| + \frac{1}{2} \beta_i^2 \right) \right), \qquad (10)$$

where $w_i = 1/|\hat{\beta}_{i,OLS}|$ and λ is selected equation-by-equation by 10-fold cross validation.

Illustrated identification strategy (i.e. order of orthogonalization)



Other possible identification strategies



→ 3 clusters means 3!=6 possible identification strategies
→ compute variance-covariance matrix for all 6 possible identification strategies
→ compute average connectedness across identification strategies

Comparison: if orthonoganlizig asset by asset

→ 15 assets implies $15! = \sim 1.3 \times 10^{12}$

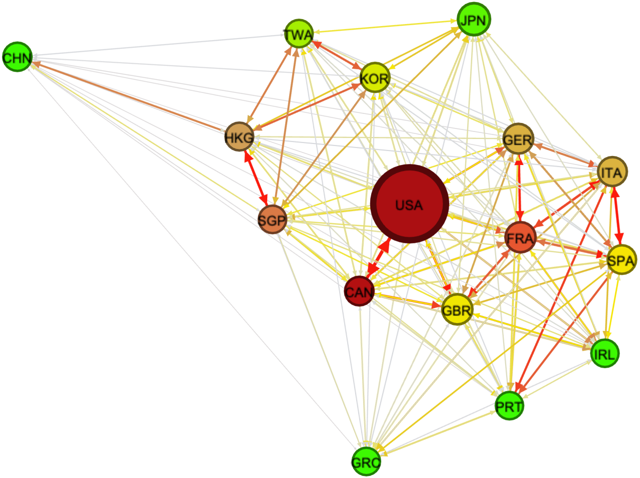
→ 20 assets implies $20! = \sim 2.4 \times 10^{18}$

– Part 4 –

Results

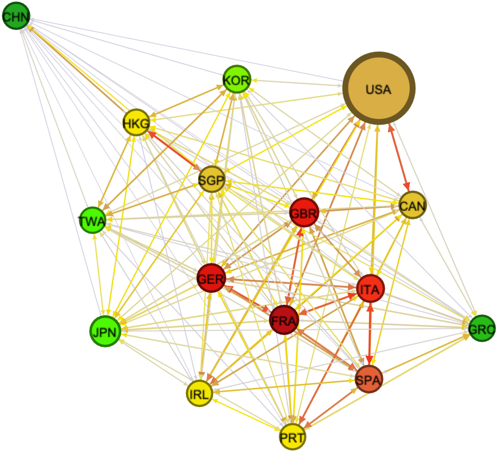
- Full Sample
- Rolling Sample
- Discussion: Feedback Loop

Network Graph, Clustered Identification



- USA at the center, biggest net emitter
- Strong regional interconnectedness

Network Graph, Generalized Identification



- USA at the outscirts
- East Asia and Europe constitute one interconnected entity

	United States	Canada	France	Germany	United Kingdom	Portugal	Ireland	Italy	Greece	Spain	Japan	China	South Korea	Taiwan	Hong Kong	Singapore
United States	1	0.817	0.7819	0.7672	0.7829	0.5784	0.6582	0.711	0.4496	0.697	0.5496	0.2443	0.5266	0.5302	0.5701	0.6058
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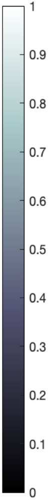
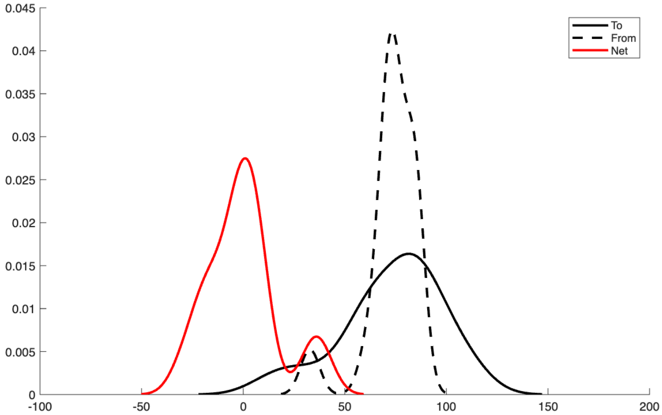
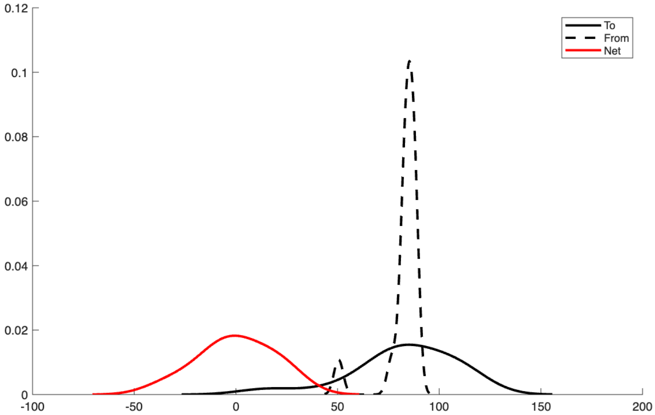


Connectedness Densities, Clustered Identification



- small group of strong net emitters

Connectedness Densities, Generalized Identification



- no apparent net emitters

Under generalized approach

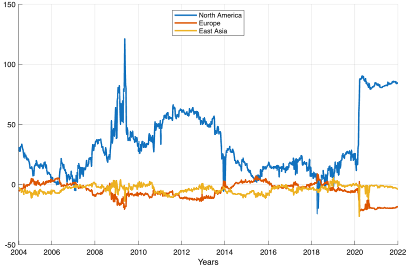
- shocks are correlated
- feedback loop of shocks

In the absence of orthogonalization across clusters

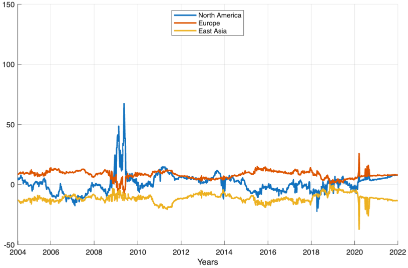
- shocks are subject to a feedback loop that smooths them across the system
- shocks cannot be properly attributed to their origin.

Net Connectedness = Shocks “to” others - Shocks “from” others

(a) Clustered Identification



(b) Generalized Identification



Clustered Approach

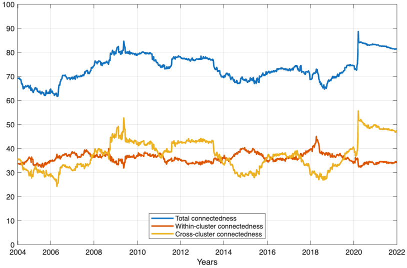
- The financial crisis of 2007-2009.
- From 2010-2014 the North American crisis moves to Europe. By 2015, series of European crises in Greece, Portugal, Ireland, Iceland, Italy, and Spain.
- The onset of the COVID-19 pandemic in 2020.

Generalized Approach

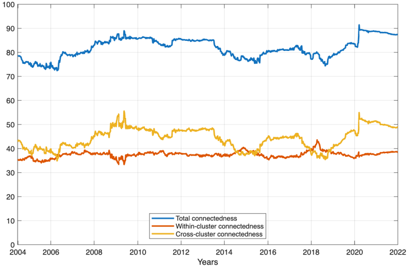
- the movements are less pronounced, particularly for North America
- The 2020 pandemic outbreak is hardly noticeable

Rolling Sample: System-Wide Connectedness

(a) Clustered Identification

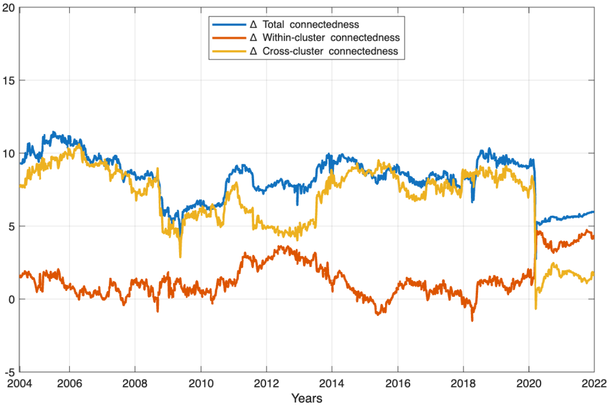


(b) Generalized Identification



Differences of Clustered and Generalized Approaches

$$\Delta = \text{ Generalized Connectedness } - \text{ Clustered Connectedness}$$



$$\Delta = \text{ Generalized Connectedness } - \text{ Clustered Connectedness}$$

The Δ is always positive:

- Most of the total difference stems from the cross-cluster difference.
- The clustered and generalized approaches can produce very different results.

Cross cluster difference

- Generalized approach captures causal *and* non-causal linkages
- Clustered approach captures *only* causal linkages

Feedback loop

$\rightarrow$ Shocks reverberate more across the system when not orthogonalized by region

$\rightarrow$ Greater cross-cluster connectedness with the generalized approach

The generalized approach

- yields network graph similar to pairwise correlation
- higher total connectedness (feedback loop)

The clustered approach

- cuts the feedback loop
- is able to attribute shocks to their origin
- establishes causal linkages across clusters
- accommodates heterogenous dynamics within clusters

– Part 5 –

Conclusion

Trade-off: causality vs. implementability

- Orthogonal IRFs from Sims (1980)
- Generalized IRFs from Koop et al. (1996)

$\rightarrow$ Cluster-orthogonalized IRFs

Application to global equity markets

- Causal linkages across regions
- Accommodate regional co-movement

In the absence of orthogonalization: Feedback loop

- Shocks reverberate across the network and smooth out
- Hides origin and causal linkages of network nodes

Thank you for your attention!

Following the approach of Koop et al. (1996) by assuming a linear form for the conditional expectation, the impulse response function is then given by

$$\boldsymbol{\Psi}^{C}(h) = \boldsymbol{A}_h \boldsymbol{Q}_C \mathbb{E}[\boldsymbol{\epsilon}_t | \boldsymbol{\epsilon}_t = \boldsymbol{\delta}] \tag{11}$$

Goal: understand the reverberation of shocks across assets.

→ only consider a shock δ_j to the j^{th} element of $\boldsymbol{\epsilon}_t$, denoted $\epsilon_{j,t}$

→ able to solely capture the effect of that shock without the effects of the other shocks occurring at the same time

The impulse response to a shock of the j^{th} element on $\boldsymbol{x}_{t+h}$ for the multivariate linear model is given by

$$\begin{aligned}\boldsymbol{\psi}_j^{C}(h) &= \frac{\boldsymbol{A}_h \boldsymbol{Q}_C \boldsymbol{\Omega}_C \boldsymbol{e}_j \delta_j}{\omega_{C,jj}} && (unscaled)\\ &= \frac{\boldsymbol{A}_h \boldsymbol{Q}_C \boldsymbol{\Omega}_C \boldsymbol{e}_j}{\sqrt{\omega_{C,jj}}} && (scaled;\ \delta_j = \sqrt{\omega_{C,jj}})\end{aligned} \tag{12}$$

where $\boldsymbol{e}_j$ is a $[N \times 1]$ selection vector which is equal to one in the j^{th} position, and zero elsewhere.

Back

In this section we derive a lower triangular matrix $\boldsymbol{Q}_C^{-1}$, such that multiplying the residuals $\boldsymbol{u}_t$ from the VAR with $\boldsymbol{Q}_C^{-1}$ orthogonalizes residuals across market segments, i.e. $\boldsymbol{u}_t = [\boldsymbol{u}_{1,t}\ \boldsymbol{u}_{2,t}\ \boldsymbol{u}_{3,t}]^\top$.

For purposes of tractability we focus on the case with $C = 3$ market segments. Mathematically,

$$\mathbb{V}[\boldsymbol{u}_t] = \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma_{11}} & \boldsymbol{\Sigma_{12}} & \boldsymbol{\Sigma_{13}} \\ \boldsymbol{\Sigma_{21}} & \boldsymbol{\Sigma_{22}} & \boldsymbol{\Sigma_{23}} \\ \boldsymbol{\Sigma_{31}} & \boldsymbol{\Sigma_{32}} & \boldsymbol{\Sigma_{33}} \end{pmatrix}$$

$$\Rightarrow \quad \mathbb{V}[\boldsymbol{Q}_3^{-1}\boldsymbol{u}_t] = \mathbb{V}[\boldsymbol{\epsilon}_t] = \boldsymbol{\Omega} = \begin{pmatrix} \boldsymbol{\Omega}_{11} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Omega}_{22} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{\Omega}_{33} \end{pmatrix}$$

where $\boldsymbol{0}$ denotes a matrix of zeros.

Back

We orthogonalize residuals across market segments with linear projections.

$$
\begin{aligned}
\boldsymbol{\epsilon}_{3,t} =& \boldsymbol{u}_{3,t} - \begin{pmatrix}\boldsymbol{\Sigma}_{31} & \boldsymbol{\Sigma}_{32}\end{pmatrix} \begin{pmatrix}\boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22}\end{pmatrix}^{-1} \begin{pmatrix}\boldsymbol{u}_{1t} \\ \boldsymbol{u}_{2,t}\end{pmatrix} \\
=& \boldsymbol{u}_{3,t} - \begin{pmatrix}\boldsymbol{\Sigma}_{31} & \boldsymbol{\Sigma}_{32}\end{pmatrix} \begin{pmatrix}\boldsymbol{\Sigma}^{11} & \boldsymbol{\Sigma}^{12} \\ \boldsymbol{\Sigma}^{21} & \boldsymbol{\Sigma}^{22}\end{pmatrix} \begin{pmatrix}\boldsymbol{u}_{1,t} \\ \boldsymbol{u}_{2,t}\end{pmatrix} \\
=& \boldsymbol{u}_{3,t} - \begin{pmatrix}\boldsymbol{\Sigma}_{31}\boldsymbol{\Sigma}^{11} + \boldsymbol{\Sigma}_{32}\boldsymbol{\Sigma}^{21} & \boldsymbol{\Sigma}_{31}\boldsymbol{\Sigma}^{12} + \boldsymbol{\Sigma}_{32}\boldsymbol{\Sigma}^{22}\end{pmatrix} \begin{pmatrix}\boldsymbol{u}_{1,t} \\ \boldsymbol{u}_{2,t}\end{pmatrix} \\
\boldsymbol{\epsilon}_{2,t} =& \boldsymbol{u}_{2,t} - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{u}_{1,t} \\
\boldsymbol{\epsilon}_{1,t} =& \boldsymbol{u}_{1,t}
\end{aligned}
$$

where $\boldsymbol{\epsilon}_{1,t}$, $\boldsymbol{\epsilon}_{2,t}$ and $\boldsymbol{\epsilon}_{3,t}$ denote the orthogonalized counterparts of $\boldsymbol{u}_{1,t}$, $\boldsymbol{u}_{2,t}$ and $\boldsymbol{u}_{3,t}$, respectively. Solving for block inverse matrix depicted in equation (5) yields

$$
\begin{pmatrix}\boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22}\end{pmatrix}^{-1} = \begin{pmatrix}\boldsymbol{\Sigma}^{11} & \boldsymbol{\Sigma}^{12} \\ \boldsymbol{\Sigma}^{21} & \boldsymbol{\Sigma}^{22}\end{pmatrix} = \begin{pmatrix}\boldsymbol{\Sigma}_{11}^{-1} + \boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}K\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1} & -\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}K \\ K\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1} & K\end{pmatrix}
$$

where $K = (\boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12})^{-1}$ is referred to as the Schur complement.

Back

Rewriting equations (6), (7) and (8) in matrix notation yields

$$\begin{aligned}
\begin{pmatrix} \boldsymbol{\epsilon}_{1,t} \\ \boldsymbol{\epsilon}_{2,t} \\ \boldsymbol{\epsilon}_{3,t} \end{pmatrix} &= \begin{pmatrix} \boldsymbol{u}_{1,t} \\ \boldsymbol{u}_{2,t} \\ \boldsymbol{u}_{3,t} \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1} & 0 & 0 \\ \boldsymbol{\Sigma}_{31}\boldsymbol{\Sigma}^{11} + \boldsymbol{\Sigma}_{32}\boldsymbol{\Sigma}^{21} & \boldsymbol{\Sigma}_{31}\boldsymbol{\Sigma}^{12} + \boldsymbol{\Sigma}_{32}\boldsymbol{\Sigma}^{22} & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_{1,t} \\ \boldsymbol{u}_{2,t} \\ \boldsymbol{u}_{3,t} \end{pmatrix} \\
&= \begin{pmatrix} \boldsymbol{I} & 0 & 0 \\ -\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1} & \boldsymbol{I} & 0 \\ -\boldsymbol{\Sigma}_{31}\boldsymbol{\Sigma}^{11} - \boldsymbol{\Sigma}_{32}\boldsymbol{\Sigma}^{21} & -\boldsymbol{\Sigma}_{31}\boldsymbol{\Sigma}^{12} - \boldsymbol{\Sigma}_{32}\boldsymbol{\Sigma}^{22} & \boldsymbol{I} \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_{1,t} \\ \boldsymbol{u}_{2,t} \\ \boldsymbol{u}_{3,t} \end{pmatrix} \qquad (13) \\
\boldsymbol{\epsilon}_t &= \boldsymbol{Q}_3^{-1}\boldsymbol{u}_t
\end{aligned}$$

Back

In parallel to the IRF equations (3) and (9), the VD equation (10) nests both orthogonalized and generalized versions:

$$\tilde{\theta}^{o}_{ij}(H) = \tilde{\theta}^{N}_{ij}(H) = \frac{\sum_{h=0}^{H-1}(\boldsymbol{e}_i'\boldsymbol{A}_h\boldsymbol{M}\boldsymbol{e}_j)^2}{\sum_{h=0}^{H-1}(\boldsymbol{e}_i'\boldsymbol{A}_h\boldsymbol{\Sigma}\boldsymbol{A}_h'\boldsymbol{e}_i)^2}$$

and

$$\tilde{\theta}^{g}_{ij}(H) = \tilde{\theta}^{1}_{ij}(H) = \frac{\sigma_{jj}^{-1}\sum_{h=0}^{H-1}(\boldsymbol{e}_i'\boldsymbol{A}_h\boldsymbol{\Sigma}\boldsymbol{e}_j)^2}{\sum_{h=0}^{H-1}(\boldsymbol{e}_i'\boldsymbol{A}_h\boldsymbol{\Sigma}\boldsymbol{A}_h'\boldsymbol{e}_i)^2}.$$

Due to the non-zero covariance of residuals in we note that $\sum_{j=1}^{N}\tilde{\theta}^{g}_{ij}(H) \neq 1$. In line with Diebold and Yilmaz (2012), we normalize to produce $\theta^{g}_{ij}(H) = \frac{\tilde{\theta}^{g}_{ij}(H)}{\sum_{j=1}^{N}\tilde{\theta}^{g}_{ij}(H)}$.

Back

Co-movement shares capture how much of the variance of the asset class C is due to co-movements among asset in that same specific asset class.

$$\Theta_m^{move} = \frac{1}{N_m} \sum_{\substack{i,j \in m \\ i \neq j}} \theta_{ij}$$

Next, the contagion share captures to which extend the VFE of asset class C is driven by shocks of k. We denote this measure by $\Theta_{m \leftarrow k}$

$$\Theta_{m \leftarrow k}^{con} = \frac{1}{N_m} \sum_{i \in m} \underbrace{\left(\frac{1}{N_k} \sum_{j \in k} \theta_{ij} \right)}_{\substack{\text{Average impact of all} \\ \text{assets in } k \text{ on asset } i \text{ in } C}}$$

The total contagion received by an asset class C is then defined as

$$\Theta_{m \leftarrow \bullet}^{con} = \sum_{k \neq m} \Theta_{m \leftarrow k}^{con}$$

Back